Journal of Information Hiding and Multimedia Signal Processing ©2015 ISSN 2073-4212
Ubiquitous International Volume **6**, Number **2**, March **2015**

Deterministic Small-World Network Extended From Triangles

Zhi-Min Fang[1], Zhe-Ming Lu[2*], Xin-Feng Li[2], Mei-Lei Lv[1], and Lu Zhang[1]

[1]College of Electrical and Information Engineering
Quzhou University
Quzhou 324000, P. R. China
fzim@21cn.com; Meilei_lv@163.com; zhanglu90573@163.com

[2]School of Aeronautics and Astronautics
Zhejiang University
Hangzhou, 310027, P.R. China
zheminglu@zju.edu.cn; lixinfeng@zju.edu.cn

Received September, 2014; revised October, 2014
(Communicated by Zhe-Ming Lu*)

Abstract. *Small-world networks are abundant in nature and society, describing such diverse systems as the World Wide Web, communication networks or the neuronal network of the worm C.elegans. In the past ten years, many small-world network models have been proposed in both probabilistic and deterministic fashions. In this letter, we propose a new deterministic small-world model by simply extending triangles. Furthermore, we analytically solve the main characteristics of the model and the results show that it is indeed a kind of small-world networks.*
Keywords: Deterministic small-world networks, Triangle-extended networks, Degree distribution, Average path length, Clustering coefficient.

---

1. **Introduction.** The study of complex networks has emerged over the past several years as a theme spanning many disciplines, ranging from mathematics and computer science to the social and biological sciences.Small-world properties have been found in a number of real-life systems. Small-world networks are famous for their small average node degree, high clustering coefficient and short average path length. The small-world phenomenon C the principle that we are all linked by short chains of acquaintances or six degrees of separation has long been the subject of anecdotal fascination among the general public, and more recently has become the subject of both experimental and theoretical research. At its most basic level, it is a statement about networks, and human social networks in particular; it concerns the graph with one node corresponding to each person in the world, and an edge joining two people if they know each other on a first-name basis. When we say that this graph is a small world, we mean, informally, that almost every pair of nodes is connected by a path with an extremely small number of steps. In the past dozen years, a considerable number of probabilistic models have been proposed. The first pioneering WS small-world model [1] was proposed by randomly rewiring each edge in a ring lattice, resulting in an avalanche of research of complex networks. One year later, another significant model called NW model [2, 3] was proposed, in which edges are randomly added between each pair of unlinked nodes, but no edges are removed from the regular lattice. Subsequently, another small-world network called R+T network [4] was

presented, which is actually a regular network coupled with a tree structure. Furthermore, a generalization of the WS model based on a two-dimensional square lattice was proposed in [5]. To find out other manners for constructing small-world networks, a simple growing evolution model based on geographical attachment preference was proposed in [6].

However, all these models mentioned above are random. As Barabási et al. said, the randomness, while in line with the major features of real-life networks, makes it harder to gain a visual understanding of how networks are shaped, and how do different nodes relate to each other [7]. Thus, many researchers turn to constructing scale-free or small-world networks in deterministic manners. In 2000, the first deterministic small-world network based on graph-theoretic methods was proposed [8]. Two years later, other two deterministic small-world models with constant and variable degree distributions were introduced [9]. Later, based on the decomposition of prime numbers, Corso constructed an interesting prime number small-world model [10]. Inspired by Corso' s work, a small-world network based on Goldbach conjecture was proposed [11]. One year later, another small-world model based on Cayley graphs was proposed [12]. In the same year, in order to get growing small-world networks, Zhang et al. proposed a deterministic small-world model by edge iterations [13]. Besides, by using the famous tower of Hanoi puzzle, Boettcher et al. proposed a fascinating deterministic model called Hanoi Network [14]. In 2012, Guo et al. proposed a tree-structured deterministic small-world network by adding edges on the binary tree [15]. Furthermore, Lu et al. proposed a small-world model derived from the deterministic uniform recursive tree with a simple adding-edge rule [16]. Shortly afterwards, an admirable deterministic scale-free small-world network model was proposed [17]. In this letter, we aim to obtain a new deterministic small-world model by simply extending a triangle.

2. **The Model.** In our scheme, the network is created by an iterative method. Assume that the obtained network after t iterations is SW_t that has N_t nodes and E_t edges. Define twin nodes as a pair of nodes that connect to each other. The proposed construction algorithm can be illustrated as follows: For $t = 0$, SW_0 is a triangle whose three nodes connect to each other. For $t \geq 1$, SW_t is obtained from SW_{t-1} by adding twin nodes for each node created at Step $t-1$ and attaching them to the node (see Fig.1).

Now we compute the size and order of SW_t. In the revolution process of the model, for each node generated at Step $t-1$, two new nodes are created at step t. Therefore,

$$N_t - N_{t-1} = 2(N_{t-1} - N_{t-2}), t > 1 \quad (1)$$

where N_t is the total number of nodes in the network. Since $N_0 = 3$ and $N_1 - N_0 = 6$, it follows that

$$N_t = 6 \times 2^t - 3 \quad (2)$$

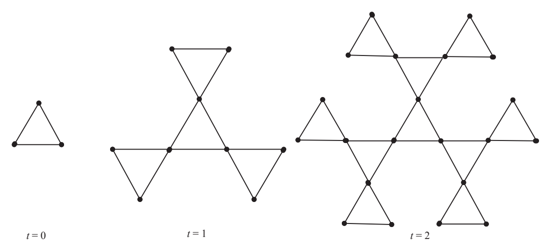

FIGURE 1. Construction of the Proposed Deterministic Small-world Network The first three steps of the iterative process are shown.

On the other hand, the addition of each new node leads to three new edges, therefore,

$$E_t - E_{t-1} = 3(N_{t-1} - N_{t-2}) \tag{3}$$

where E_t is the total number of edges in the network. As $E_0 = 3$ and $E_1 - E_0 = 9$, we have

$$E_t = 9 \times 2^t - 6 \tag{4}$$

Thus, the average node degree can be calculated as follows

$$\langle k \rangle_t = \frac{2E_t}{N_t} = \frac{2(9 \times 2^t - 6)}{6 \times 2^t - 3} = 3 - \frac{1}{2^{t+1} - 1} \tag{5}$$

Since $\lim_{t\to\infty} \langle k \rangle_t = 3$,we can see that the proposed network is a sparse graph whose nodes have many fewer connections than possible. Compared with our model, the average node degree of the deterministic model presented in Ref.[13] approaches 4.

3. **Topological Properties.** Thanks to its deterministic and discrete nature, the model proposed above can be solved exactly. To show its small-world nature, in the following three subsections, we concentrate on its degree distribution, clustering coefficient and diameter.

3.1. **Degree Distribution.** The degree distribution is one of the most important statistical characteristics of a network. In the study of graphs and networks, the degree of a node in a network is the number of connections it has to other nodes and the degree distribution is the probability distribution of these degrees over the whole network. The degree of Node i can be defined as the number of its direct connections to other nodes, and the degree distribution $P(k)$ is defined as the probability that a randomly selected node has exactly k links. According to the iterative algorithm, for $t > 0$, we can easily prove that the possible degree values in the proposed network are 2 and 4, which correspond to the nodes created at Step t (called the outer layer nodes) and the other nodes (called the inner layer nodes) respectively. Thus, we can easily obtain

$$P(k) = \frac{6 \times 2^{t-1}}{6 \times 2^t - 3}\delta(k-2) + \frac{6 \times 2^{t-1} - 3}{6 \times 2^t - 3}\delta(k-4) \tag{6}$$

Where $\delta(k) = 1$ for $k = 0$ and $\delta(k) = 0$ for $k \neq 0$. When $t \to \infty$, we can easily obtain the following degree distribution

$$\lim_{t\to\infty} P(k) = 0.5 \times \delta(k-2) + 0.5 \times \delta(k-4) \tag{7}$$

Therefore, the degree distribution of the proposed network is discrete and focuses mainly on two degree values. This is usually called delta distribution, which is different from most proposed small-world models including the WS model.

3.2. **Clustering Coefficient.** Clustering is another important property of a network, which provides a measure of degree to which nodes in a graph tend to cluster together. Roughly speaking it tells how well connected the neighborhood of the node is. If the neighborhood is fully connected, the clustering coefficient is 1 and a value close to 0 means that there are hardly any connections in the neighborhood. The most immediate measure of clustering is the clustering coefficient C_i for every node i, which is defined as the number of links L_i that actually exist between its nearest neighbors divided by the number of links that could possibly exist between them, i.e., $C_i = 2L_i/k_i(k_i - 1)$. The clustering coefficient of the whole network is obtained by averaging over all individual C_i. The global clustering coefficient can be also defined using triplets of nodes. A triplet is three nodes that are connected by either two (open triplet) or three (closed triplet) undirected ties. A triangle consists of three closed triplets, one centred on each of the nodes. The global clustering coefficient is the number of closed triplets (or 3 × triangles) over the total number of triplets (both open and closed). Here, because of the symmetry of the proposed network, the nodes with the same degree k have the same local clustering coefficient $C(k)$. For $t > 0$, we can easily prove that the local clustering coefficient of outer layer nodes is 1, while the local clustering coefficient of inner layer nodes is 1/3. Thus, we can easily get the following results

$$C(k) = 1 \times \delta(k-2) + \frac{1}{3} \times \delta(k-4) \tag{8}$$

Furthermore, the average clustering coefficient $\langle C \rangle$ can be easily obtained as follows.

$$\begin{aligned} \langle C \rangle &= \frac{1}{6 \times 2^t - 3} \sum_{i=1}^{6\times 2^t - 3} C_i(t) \\ &= \frac{1}{6 \times 2^t - 3} \left[(6 \times 2^{t-1}) \times 1 + (6 \times 2^{t-1} - 3) \times \frac{1}{3} \right] \\ &= \frac{2}{3} + \frac{1}{3(2^{t+1} - 1)} \end{aligned} \tag{9}$$

It can be seen that the clustering coefficient $\langle C \rangle$ decreases monotonically with the iteration step t. When t approaches infinity, we get the minimum value $\langle C \rangle = 2/3 = 0.6667$, so the clustering coefficient of the proposed network is high.

3.3. **Diameter and Average Path Length.** In the mathematical field of graph theory, the distance between two vertices in a graph is the number of edges in a shortest path (also called a graph geodesic) connecting them. This is also known as the geodesic distance. The diameter of the network is defined as the longest geodesic, i.e., the longest shortest path between any two nodes in the network. Average path length is a concept in network topology that is defined as the average number of steps along the shortest paths for all possible pairs of network nodes. It is a measure of the efficiency of information or mass transport on a network. The small-world concept describes the fact that there is a relatively short distance between most pairs of nodes compared with the size of network. The average path length (APL) well characterizes this feature, which is defined as the average distance over all possible pairs of nodes. In general, it is hard to obtain the analytic solution of APL, thus researchers adopt another parameter called diameter to demonstrate the short distance between any two nodes in a network. By definition, diameter is the maximal distance between any pair of nodes, which characterizes the maximal communication delay in a network. If a network is with a small diameter, it is undoubtedly with a short APL. Denote the diameter at Iteration t as $D(t)$. Because of the symmetry of the proposed network, we can easily prove that the diameter always lies between the new added nodes, and for each iteration the diameter always increases by 2 on the basis of the previous iteration. Therefore, we can get the following result

$$D(t) = D(t-1) + 2, t > 0 \tag{10}$$

Since $D(0) = 1$, according to Eq. (2), we have

$$D(t) = 2t + 1 = 2\log_2 \frac{N_t + 3}{3} - 1 \tag{11}$$

Thus, the diameter grows logarithmically with the number of nodes. Because the average path length is smaller than D, so the APL should increase more slowly. To show the relationship more clearly, we give the simulation results in Fig.2.

Based on the above discussions, we can conclude that the proposed model is a deterministic small-world network, because it is a sparse one with high clustering and short

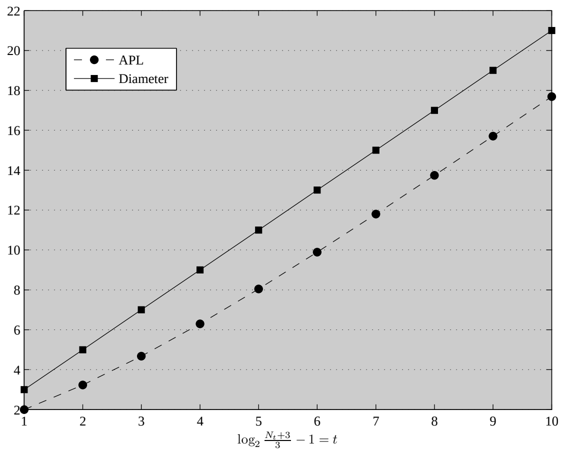

FIGURE 2. The APL and Diameter Versus the Logarithm of the Number of Nodes.

diameter and average path length, which satisfy the three main necessary features for small-world networks.

4. **Conclusions.** In this letter, we have proposed a deterministic small-world model by simply extending a triangle. We have derived the analytic solutions for degree distribution, clustering coefficient and diameter of the network, and they are all close to those for existing random small-world networks. The proposed model may provide a new way to generate a network with specific properties and help to understand some properties of real-world planar networks. Future work will focus on finding the small-world phenomenon in image processing and image analysis[18, 19, 20, 21].

Acknowledgments. This work is supported by the National Natural Science Foundation of China under grant 61171150, and also supported by the program of the 12th five-year-plan key discipline (Detection technology and automatic equipment) of Zhejiang Province, China. The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

REFERENCES

[1] D. J. Watts, and S. H. Strogatz, Collective dynamics of small-world networks, *Nature*, vol. 393, no. 6684, pp. 440-442, 1998.
[2] M. E. Newman, and D. J. Watts, Renormalization group analysis of the small-world network model, *Physics Letters A*, vol. 263, no. 4, pp. 341-346, 1999.
[3] M. E. Newman, Scientific collaboration networks. i. network construction and fundamental results, *Physical review E*, vol. 64, no. 1, pp. 016131, 2001.
[4] R. Kasturirangan, Multiple scales in small-world graphs, arXiv preprint cond-mat/9904055, 1999.
[5] J. M. Kleinberg, Navigation in a small world, *Nature*, vol. 406, no. 6798, pp. 845-845, 2000.
[6] J. Ozik, B. R. Hunt, and E. Ott, Growing networks with geographical attachment preference: Emergence of small worlds, *Physical Review E*, vol. 69, no. 2, pp. 026108, 2004.
[7] A. L. Barabási, E. Ravasz, and T. Vicsek, Deterministic scale-free networks, *Physica A: Statistical Mechanics and its Applications*, vol. 299, no. 3, pp. 559-564, 2001.
[8] F. Comellas, J. Ozon, and J. G. Peters, Deterministic small-world communication networks, *Information Processing Letters*, vol. 76, no. 1, pp. 83-90, 2000.
[9] F. Comellas, and M. Sampels, Deterministic small-world networks,*Physica A: Statistical Mechanics and its Applications*, vol. 309, no. 1, pp. 231-235, 2002.

[10] G. Corso, Families and clustering in a natural numbers network,*Physical Review E*, vol. 69, no. 3, pp. 036106, 2004.

[11] A. K. Chandra, and S. Dasgupta, A small world network of prime numbers,*Physica A: Statistical Mechanics and its Applications*, vol. 357, no. 3-4, pp.436-446, 2005.

[12] W. Xiao, and B. Parhami, Cayley graphs as models of deterministic small-world networks, *Information Processing Letters*, vol. 97, no. 3, pp. 115-117, 2006.

[13] Z. Zhang, L. Rong, and C. Guo, A deterministic small-world network created by edge iterations,*Physica A: Statistical Mechanics and its Applications*, vol. 363, no. 2, pp. 567-572, 2006.

[14] S. Boettcher, B. Gonçalves, and H. Guclu, Hierarchical regular small-world networks,*ournal of Physics A: Mathematical and Theoretical*, vol. 41, no. 25, pp. 252001, 2008.

[15] S. Z. Guo, Z. M. Lu, Z. Chen, and H. Luo, A tree-structured deterministic small-world network, *IEICE Trans. on Information and Systems*, vol. 95, no. 5, pp. 1536-1538, 2012.

[16] Z. M. Lu, and S. Z. Guo, A small-world network derived from the deterministic uniform recursive tree, *Physica A: Statistical Mechanics and its Applications*, vol. 391, no.1, pp. 87-92, 2012.

[17] Z. M. Lu, Y. X. Su, and S. Z. Guo, Deterministic scale-free smallworld networks of arbitrary order, *Physica A: Statistical Mechanics and its Applications*, vol. 392, no. 17, pp. 3555-3562, 2013.

[18] X. F. Liu, and X. Zhu, Kernel-optimized based Fisher classification of Hyperspectral imagery, *Journal of Information Hiding and Multimedia Signal Processing*, vol. 5, no. 3, pp. 431-438, 2014.

[19] X. X. Wang, and L. Y. Ma, A compact K nearest neighbor classification for power plant fault diagnosis, *Journal of Information Hiding and Multimedia Signal Processing*, vol. 5, no. 3, pp. 508-517, 2014.

[20] T. C. Lin, H. C. Huang, B. Y. Liao, and J. S. Pan, An optimized approach on applying genetic algorithm to adaptive cluster validity index, *International Journal of Computer Sciences and Engineering Systems*, vol. 1, no. 4, pp. 253-257, 2007.

[21] M. S. Lai, H. C. Huang, S. C. Chu, Y. H. Huang, and K. C. Huang, Image texture segmentation with ant colony systems, *International Conference on Innovative Computing, Information and Control*, vol. 1, pp. 652-655, 2006.