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On Rainbow Vertex Antimagic Coloring of Amalgamation Graph

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Abstract. *The graph topic of this study is Rainbow Vertex Antimagic Coloring (RVAC), which combines the notions of Rainbow Vertex Connection and Antimagic Labeling. This work establishes six theorems: three on the rainbow vertex connection number and three on the rainbow vertex antimagic connection number for the Amalgamation Cuttlefish graph* $Amal(CF_r, z, s)$*, the Amalgamation Semi Jellyfish graph* $Amal(SJ_r, z, s)$*, and the Amalgamation Diamond graph* $Amal(D_r, z, s)$*. By employing a bijective edge labeling function to induce vertex weights, we demonstrate vertex rainbow connectivity and construct an antimagic labeling that yields rainbow paths between any two vertices. The main results provide the exact values of rvc and rvac for both graphs, namely* $rvc(Amal(CF_r, z, s)) = s+1$ *(for* $s \geq 3$ *,* $r \geq 3$*),* $rvac(Amal(CF_r, z, s)) = sr+1$ *(for* $s \geq 3$*,* $r \geq 3$*),* $rvc(Amal(SJ_r, z, s)) = s+1$ *(for* $s \geq 3$*,),* $rvac(Amal(SJ_r, z, s)) = sr+1$ *(for* $s \geq 3, r \geq 3$*),* $rvc(Amal(D_r, z, s)) = 3$ *(for* $s \geq 2$*,* $r \geq 2$*) and* $rvac(Amal(D_r, z, s)) = 4$ *(for* $s \geq 2$*,* $r \geq 2$*). The method is deductive constructive, involving explicit edge-label constructions, vertex-weight computations, and verification of representative rainbow paths.*

Keywords: Rainbow Vertex Connection Number, Rainbow Vertex Antimagic Coloring, Amalgamation of Graph.

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1. **Introduction.** Graph theory has become an important branch of modern mathematics due to its solid theoretical basis and its broad use in representing relational structures. A graph G consists of two sets, namely $V(G)$ as the vertex set and $E(G)$ as the edge set, where each edge joins two vertices u and v with $u, v \in V(G)$ [1]. Accordingly, a graph is denoted by $G(V, E)$. In general, the basic structure of a graph is described by its order and size [2]. Thus, every graph has at least one vertex, although it may contain no edges [3]. Among the central themes in graph theory is connectivity, which has generated many significant and elegant results [4]. Another major topic is graph coloring, a subject that continues to attract considerable research attention [5]. In this regard, connectivity parameters associated with coloring are particularly meaningful, since they show how graph structures influence the existence of certain connecting patterns.

Graph coloring is one of the classical and well-established topics in graph theory. Since the introduction of graph theory by Leonhard Euler in 1736, many developments have emerged in this field, including rainbow coloring [6]. Several forms of rainbow coloring have been studied, such as rainbow edge coloring, rainbow vertex coloring, and strong rainbow edge or vertex coloring [7]. In rainbow coloring, the essential requirement is the existence of a rainbow path between every pair of vertices u and v, where the internal vertices of the path receive pairwise distinct colors [8]. This idea is important because it connects local coloring rules with global connectivity properties. As a result, rainbow-type coloring parameters have become a productive area of study, especially in efforts to determine exact values for particular classes of graphs.

The concept of rainbow connectivity was initially proposed in [9]. A graph G is said to be rainbow connected if for every two vertices u and v, there exists at least one rainbow path joining them [10, 11]. Later, this concept was extended in [12, 13] into rainbow edge-connection and rainbow vertex-connection. In another direction, antimagic labeling was introduced by Hartsfield and Ringel in 1990 [14]. In graph theory, labeling refers to assigning non-negative integers to graph elements [15]. More specifically, graph labeling assigns natural numbers in such a way that different vertices or edges may produce distinct weights. In antimagic labeling, the integers $\{1, 2, 3, \ldots, q\}$ are assigned to the edges of a graph so that all induced vertex weights are different. Although rainbow connectivity and antimagic labeling were developed from different perspectives, both are closely related through the principle of distinctness, namely distinct colors on connecting paths and distinct weights induced by labels.

A number of fundamental concepts in graph theory provide the groundwork for the study of rainbow vertex antimagic coloring, particularly those related to paths, distances, and spanning subgraphs. In a connected graph, a path consists of a sequence of neighboring vertices, whereas the shortest path connecting two vertices is referred to as a geodesic [16, 17]. The distance between two vertices u and v, written as $d(u, v)$, is defined as the length of that geodesic, and a graph is said to be connected if each pair of vertices is linked by at least one path [18]. The degree of a vertex, denoted by $\deg(v)$, represents the number of edges attached to that vertex, and a vertex with degree one is known as a pendant vertex. A graph that contains no cycles is termed a tree, while a spanning tree is a tree that includes every vertex of the original graph [16]. Moreover, in proper vertex coloring, any two adjacent vertices must be assigned distinct colors, and the smallest number of colors required is known as the chromatic number, denoted by $\chi(G)$ [19]. These classical ideas provide the theoretical basis for rainbow vertex antimagic coloring, since the vertex weights arising from antimagic labeling may also be interpreted as colors that support rainbow vertex connectivity [12]. In particular, geodesic structure, vertex degrees, and pendant configurations strongly affect whether a labeling-induced coloring can satisfy rainbow connectivity conditions.

Rainbow vertex antimagic coloring merges two important notions, namely rainbow vertex connection and antimagic labeling [20]. A function f is called an antimagic edge labeling if it induces distinct weights on all vertices. A path P in an edge-labeled graph G is called rainbow if, for any pair of vertices u and v, the internal vertices along the $u - v$ path possess pairwise distinct induced weights [11]. Likewise, rainbow vertex coloring requires that the internal vertices on every relevant path between u and v have pairwise distinct colors [21]. The rainbow vertex antimagic connection number of a graph G, written as $rvac(G)$, is defined as the smallest number of colors taken over all rainbow colorings produced by rainbow vertex antimagic labelings of G [22]. Equivalently, $rvac(G)$ represents the least number of colors required among all rainbow vertex antimagic colorings arising from a rainbow vertex antimagic labeling of G [23]. This study lies at the intersection of graph labeling and graph coloring. Antimagic labeling remains a central topic in labeling theory, while rainbow vertex coloring is an important direction in coloring theory [24]. The importance of this combined concept comes from its capacity to integrate label-induced distinctness and path-related color distinctness into a single framework, which makes the associated parameter structurally rich and difficult to determine precisely.

Moreover, vertex antimagic edge labeling refers to a bijective labeling of the edges of a graph, in which the weight of each vertex is determined by adding the labels of all edges incident to that vertex [25]. This labeling is said to be antimagic whenever the resulting vertex weights are pairwise different [26]. When the induced weights are then used as vertex colors in a way that produces rainbow connectivity, the resulting concept is called rainbow vertex antimagic coloring [27]. The minimum number of colors that can be attained in this context is called the rainbow vertex antimagic connection number, denoted by $rvac(G)$ [28]. Although this topic has drawn increasing interest, the available studies still focus mainly on introducing the concept and examining certain individual graph classes. By contrast, many modular graph families with repeated local patterns have not yet been thoroughly investigated. This situation motivates the present work, since repeated constructions often generate nontrivial connectivity behavior and distinctive weight distributions, making them promising objects for examining the interaction between antimagic labelings and rainbow vertex paths.

This paper studies three graph classes, namely $Amal(CF_r, z, s)$, $Amal(SJ_r, z, s)$, and $Amal(D_r, z, s)$. The graph $Amal(CF_r, z, s)$ is an amalgamation cuttlefish graph obtained by identifying the central vertex of several repeated cuttlefish blocks, where each block contains two internal connecting vertices together with several pendant extensions. The graph $Amal(SJ_r, z, s)$ is an amalgamation semi jellyfish graph formed by merging several semi jellyfish blocks at a common central vertex, with each block consisting of an internal supporting structure and groups of pendant vertices. Meanwhile, $Amal(D_r, z, s)$ is an amalgamation diamond graph obtained by joining several diamond blocks through one shared central vertex. In this work, we construct explicit coloring and labeling schemes in order to determine the exact values of the rainbow vertex connection number together with the rainbow vertex antimagic connection number for these graph families. The novelty of this study lies in determining the exact values of rvc and $rvac$ for these specific amalgamation graph classes, which, as far as we know, have not previously been investigated in the context of rainbow vertex antimagic coloring. Unlike previous studies that usually treat rainbow vertex connection and antimagic labeling as separate topics, this paper combines both within a unified RVAC framework and applies the approach to modular graphs with repeated structures. The main contributions are as follows: first, determining the exact connection parameters for $Amal(CF_r, z, s)$, $Amal(SJ_r, z, s)$, and $Amal(D_r, z, s)$; second, constructing explicit antimagic edge labelings together with the induced vertex-weight colorings; and third, verifying representative rainbow vertex paths to show that the obtained bounds are sharp. These findings contribute to the theoretical development of rainbow vertex antimagic coloring and open opportunities for further study on other amalgamation graph classes.

2. Methods.

The study employs a deductive analytical approach with several key stages. First, we specify the graphs by defining the vertex and edge sets of $Amal(CF_r, z, s)$, $Amal(SJ_r, z, s)$ and $Amal(D_r, z, s)$, and determining their cardinalities. A semi jellyfish graph, denoted by SJ_s, is a connected graph consisting of one central vertex z, two intermediate vertices x_1 and x_2, two upper vertices y_1 and y_2, and s pendant vertices $z_1, z_2, \ldots, z_s$, where $s \geq 1$. The central vertex z is adjacent to x_1 and x_2, while x_1 and

x_2 are also adjacent to each other. Furthermore, x_1 is adjacent to y_1, x_2 is adjacent to y_2, and y_1 is adjacent to y_2. In addition, the central vertex z is directly connected to each pendant vertex z_t, for $1 \leq t \leq s$. Hence, the subgraph formed by the vertices $\{x_1, x_2, y_1, y_2\}$ together with their connections to z represents the upper body of the graph, while the pendant vertices $z_1, z_2, \ldots, z_s$ attached to z form the tentacles. Since this structure resembles a jellyfish graph in a simpler form, it is called a semi jellyfish graph.

A cuttlefish graph, denoted by CF_s, is a connected graph consisting of one central vertex z, two lateral vertices $x_{1,1}$ and $x_{2,1}$, one upper vertex y_1, and n pendant vertices $z_{1,1}, z_{1,2}, \ldots, z_{1,s}$, where $s \geq 1$. The central vertex z is adjacent to both $x_{1,1}$ and $x_{2,1}$, while $x_{1,1}$ and $x_{2,1}$ are also adjacent to each other. Moreover, the upper vertex y_1 is adjacent to both $x_{1,1}$ and $x_{2,1}$, forming the upper body of the graph. In addition, the central vertex z is directly connected to each pendant vertex $z_{1,t}$ for $1 \leq t \leq s$, which represent the tentacles. Since the resulting structure resembles a cuttlefish, this graph is called a cuttlefish graph.

Second, we establish lower bounds by exploiting the graph diameter and local structure (in particular, the presence of star subgraphs) to lower bounds for *rvc* and *rvac*. Pattern recognition is used as a research method to identify and analyze latent regularities that are not immediately observable in the data [21]. Next, we construct the required colorings and labelings: (i) a vertex coloring that induces rainbow paths, yielding the value of *rvc*; and (ii) a bijective edge labeling that produces distinct vertex weights while simultaneously guaranteeing rainbow paths, thereby yielding the value of *rvac*. We then verify the paths by presenting representative rainbow-path tables for each pair of vertex classes. Finally, we demonstrate tightness by showing that the derived upper bounds meet the lower bounds, which establishes the exact values of *rvc* and *rvac*.

In recent years, there were several researchers who had studied rainbow vertex antimagic coloring. Among them are Marsidi et. al [20]. Some results are presented into observation and Lemma. Their lemma, we need to proof in this paper.

Lemma 1. *Let G be a connected graph. Let $rvc(G)$, $diam(G)$ be respectively the rainbow vertex connection number and the diameter of a graph, then $rvc(G) \geq diam(G) - 1$* [20].

Lemma 2. *Let G be a connected graph with no pendant vertex. Let $rvc(G)$ is the rainbow vertex antimagic connection number, then $rvac(G) \geq rvc(G)$* [20].

3. **Research Findings.** In this section, we present new results on the rainbow vertex connection number and the rainbow vertex antimagic connection number for the amalgamation graphs considered in this paper. Specifically, Theorems 1, 3, and 5 determine the exact values of the rainbow vertex connection number, whereas Theorems 2, 4, and 6 determine the exact values of the rainbow vertex antimagic connection number. These results are presented successively for the amalgamation cuttlefish graph, amalgamation semi jellyfish graph, and amalgamation diamond graph.

Lemma 3. *Let G be a connected graph. Let $rvac(G)$ be the rainbow vertex antimagic connection number of G, $P(G)$ be the set of pendant vertices of G. Then* $\mathrm{rvac}(G) \geq \max\{\mathrm{rvc}(G), |P(G)|\}$ *where $P(G) = \{v \in V(G) | deg(v) = 1\}$.*

Proof. It is known that $\mathrm{rvac}(G) \geq \mathrm{rvc}(G)$. Let $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$ be a bijective antimagic labeling of G. For each vertex $v \in V(G)$, define its induced vertex weight by

$$w(v) = \sum_{uv \in E(G)} f(uv),$$

where the summation is taken over all edges incident to v.

Since each pendant vertex is incident with exactly one edge, say uu_1 for u and vv_1 for v, we have

$$w(u) = f(uu_1) \quad \text{and} \quad w(v) = f(vv_1).$$

Since $u \neq v$, it follows that $uu_1 \neq vv_1$. Moreover, since f is bijective, distinct edges receive distinct labels. Hence,

$$f(uu_1) \neq f(vv_1),$$

which implies

$$w(u) \neq w(v).$$

Therefore, every pair of distinct pendant vertices receives distinct colors. It follows that

$$\text{rvac}(G) \geq |P(G)|.$$

Finally, we consider two cases. If $\text{rvc}(G) \geq |P(G)|$, then we use the inequality $\text{rvac}(G) \geq \text{rvc}(G)$ from Lemma 2. On the other hand, if $|P(G)| \geq \text{rvc}(G)$, then we use the result $\text{rvac}(G) \geq |P(G)|$ obtained above. Therefore, in both cases, we conclude that

$$\text{rvac}(G) \geq \max\{\text{rvc}(G), |P(G)|\}.$$

□

Theorem 1. *Let $Amal(CF_r, z, s)$ be an amalgamation of cuttlefish graph with $s \geq 3$, $r \geq 3$, the rainbow vertex connection number of $Amal(CF_r, z, s)$ is $rvc\ (Amal(CF_r, z, s)) = s + 1$.*

Proof. Let $Amal(CF_r, z, s)$ with vertex set $V(Amal(CF_r, z, s)) = \{x_{1,t} \mid 1 \leq t \leq s\} \cup \{x_{2,t} \mid 1 \leq t \leq s\} \cup \{z_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{y_t \mid 1 \leq t \leq s\} \cup \{z\}$ and edge set $E(Amal(CF_r, z, s)) = \{zx_{1,t} \mid 1 \leq t \leq s\} \cup \{zx_{2,t} \mid 1 \leq t \leq s\} \cup \{x_{1,t}x_{2,t} \mid 1 \leq t \leq s\} \cup \{x_{1,t}y_t \mid 1 \leq t \leq s\} \cup \{x_{2,t}y_t \mid 1 \leq t \leq s\} \cup \{zz_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\}$. Based on these vertex and edge sets, the cardinalities are $|V(Amal(CF_r, z, s))| = sr + 3s + 1$ and $|E(Amal(CF_r, z, s))| = sr + 5s$.

Next, we will prove that $rvc(Amal(CF_r, z, s)) = s + 1$ by showing the lower bound and upper bounds. First, we will prove the lower bound. To prove the lower bound, we use Lemma 1, which states the $rvc(G) \geq diam(G) - 1$. Since $(Amal(CF_r, z, s))$ is a connected graph, and the diameter of the graph $diam(Amal(CF_r, z, s))$ is 4, by Lemma 1 we can conclude $rvc(Amal(CF_r, z, s)) \geq diam(Amal(CF_r, z, s)) - 1 = 4 - 1 = 3$. This gives a general lower bound $rvc(Amal(CF_r, z, s)) \geq 3$.

However, we can obtain a stronger lower bound by further analyzing the graphs structure and the longest paths in it. Let us consider pairs of vertices y_t and y_{t+1} for $1 \geq t \geq s$. Each path connecting such a pair must pass through several internal vertices, which forms the longest path as follows $y_t \rightarrow x_{1,t} \rightarrow z \rightarrow x_{1,t+1} \rightarrow y_{t+1}$. Since this path must be a rainbow path (each internal vertex must have a different color), we need at least $s + 1$ distinct colors to ensure that the paths are correctly connected.

By considering all adjacent branches for each $t = 1, 2, \ldots, s$, we observe that the vertices $y_1, y_2, \ldots, y_s$, the vertices $x_{1,1}, x_{1,2}, \ldots, x_{1,s}$, the vertices $x_{2,1}.x_{2,2}, \ldots, x_{2,s}$, and the central vertex z must all receive different colors. Therefore, at least $s + 1$ colors are required to color this graph. Thus, we obtain the stronger lower bound is $rvc(Amal(CF_r, z, s)) \geq s + 1$.

Second, to prove the upper bound of $rvc(Amal(CF_r, z, s)) \leq s + 1$ by defining the vertex function as follows.

$$f(y_t) = f(z_{t,q}) = f(z) = 1\ ;\ 1 \leq t \leq s,\ 1 \leq q \leq r,$$
$$f(x_{1,t}) = f(x_{2,t}) = s + t + 1\ ;\ 1 \leq t \leq s.$$

We assume C_1 as the color set on $y_t, z_{t,q}$, and z so that $C_1 = \{1\} \Longrightarrow |C_1| = 1$. Next, we assume C_2 as the color set of the vertices $x_{1,t}$ and $x_{2,t}$ so that $C_2 = \{2, 3, \cdots, s+1\} \Longrightarrow |C_2| = s$. Based on this, the total color cardinality is $|C_1| + |C_2| = s + 1$. Then, we have $rvc(Amal(CF_r, z, s)) \leq s + 1$. Based on the lower and the upper bound, we have $s + 1 \leq rvc(Amal(CF_r, z, s)) \leq s + 1$. Then, it concludes that $rvc(Amal(CF_r, z, s)) = s + 1$. □

We observe that the upper bound $rvc(Amal(CF_r, z, s)) \leq s + 1$ is obtained from the precise allocation of colors to the vertices of the graph. This upper bound is a sharp bound, as it takes into account the structure of the graph and the fact that only $s + 1$ colors are necessary to ensure the correct rainbow vertex connection.

The rainbow path of the graph $Amal(CF_r, z, s)$ can be seen in Table 1 and an illustration of the rainbow vertex coloring of $Amal(CF_r, z, s)$ see in Figure 1.

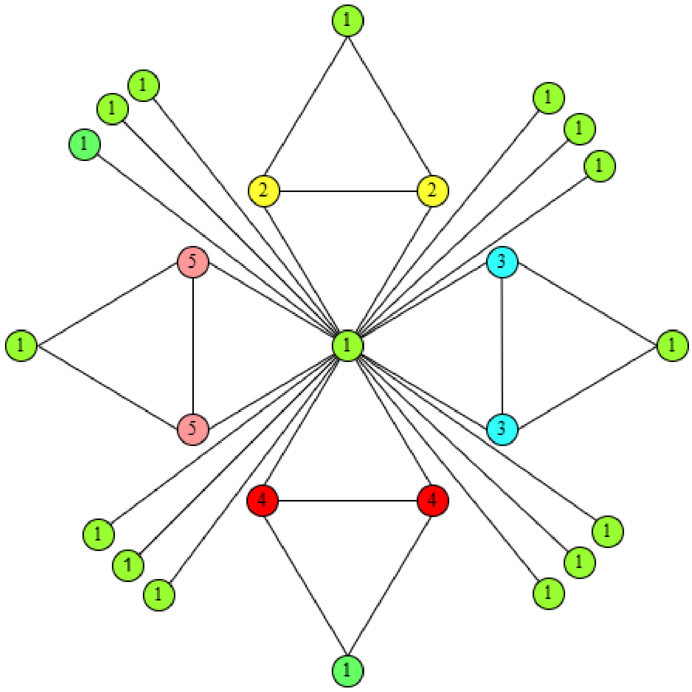

FIGURE 1. rvc of graph $Amal(CF_3, z, 4)$.

Based on Table 1 every u–v path is a rainbow path, so each path in the graph $Amal(CF_r, z, s)$ has distinct colors on its interior vertices. Then, based on Figure 1, the graph $Amal(CF_r, z, s)$ satisfies the rainbow vertex connection property. Consequently, the previously obtained lower and upper bounds coincide, and hence $rvc\big(Amal(CF_r, z, s)\big) = s + 1$, as claimed $rvc(Amal(CF_r, z, s)) = s + 1$.

Theorem 2. *Let $Amal(CF_r, z, s)$ be an amalgamation cuttlefish graph with $s \geq 3, r \geq 3$, the rainbow vertex antimagic connection number of $Amal(CF_r, z, s)$ is rvac $(Amal(CF_r, z, s)) = sr + 1$.*

Proof. Let $Amal(CF_r, z, s)$ with vertex set $V(Amal(CF_r, z, s)) = \{x_{1,t} \mid 1 \leq t \leq s\} \cup \{x_{2,t} \mid 1 \leq t \leq s\} \cup \{z_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{y_t \mid 1 \leq t \leq s\} \cup \{z\}$ and edge set $E(Amal(CF_r, z, s)) = \{zx_{1,t} \mid 1 \leq t \leq s\} \cup \{zx_{2,t} \mid 1 \leq t \leq s\} \cup \{x_{1,t}x_{2,t} \mid 1 \leq t \leq s\} \cup \{x_{1,t}y_t \mid 1 \leq t \leq s\} \cup \{x_{2,t}y_t \mid 1 \leq t \leq s\} \cup \{zz_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\}$. Based on these vertex and edge sets, the cardinalities are $|V(Amal(CF_r, z, s))| = sr + 3s + 1$ and $|E(Amal(CF_r, z, s))| = sr + 5s$.

Next, we will prove that $rvac(Amal(CF_r, z, s)) = rs + 1$ by showing the lower bound and upper bounds. First, we will prove the lower bound. To prove the lower bound, we use to Lemma 3, which states that for a connected graph G, we have $rvac(G) \geq max\{rvc(G), |P(G)|\}$, where $rvc(G)$ is the rainbow vertex connection number of graph G, and $P(G)$ is the set of pendant vertices of G. For the graph $Amal(CF_r, z, s)$, we know that the rainbow vertex connection number $rvc(Amal(CF_r, z, s)) = s + 1$ from the previous results (Theorem 1). Thus, we have $rvac(Amal(CF_r, z, s)) \geq max\{s + 1, |P(Amal(CF_r, z, s))|\}$.

We calculate $|P(Amal(CF_r, z, s))|$. The graph $Amal(CF_r, z, s)$ contains sr pendant vertices, specifically the vertices $z_{t,q}$ for $1 \leq t \leq s$ and $1 \leq q \leq r$, each of which is only connected to the central vertex z. Therefore, the number of pendant vertices $|P(Amal(CF_r, z, s))$ is sr. Thus, we obtain $rvac(Amal(CF_r, z, s)) \geq max\{s + 1, sr\}$. Since $sr \geq s+1$ for all $s \geq 3$ and $r \geq 3$ we can conclude that $rvac(Amal(CF_r, z, s)) \geq sr$. Since $|P(G)| \geq rvc(G)$, we use the inequality for $rvac(G) \geq |P(G)|$ on this graph according to Lemma 3.

However, to ensure the rainbow vertex antimagic coloring on this graph, we need to assign an additional color for the central vertex z, contributing one extra color. Therefore, the total number of colors required for the coloring is at least $sr+1$, consisting of sr colors for the pendant vertices and one color for the central vertex. Thus, we obtain the lower bound for $rvac(Amal(CF_r, z, s))$ is $rvac(Amal(CF_r, z, s)) \geq sr + 1$.

Second, prove the upper bound of $rvac(Amal(CF_r, z, s)) \leq sr+1$ by defining the vertex function as follows.

$$\begin{aligned}
f(x_{1,t}\, x_{2,t}) &= t\ ;\ 1 \leq t \leq s \\
f(x_{2,t}\, z) &= s + t\ ;\ 1 \leq t \leq s \\
f(x_{1,t}\, y_t) &= 2s + t\ ;\ 1 \leq t \leq s \\
f(x_{1,t}\, z) &= 3s + t\ ;\ 1 \leq t \leq s \\
f(x_{2,t}\, y_t) &= 4s + t\ ;\ 1 \leq t \leq s \\
f(z\, z_{t,q}) &= 5s + tr + q - r\ ;\ 1 \leq t \leq s,\ 1 \leq q \leq r
\end{aligned}$$

According to the labeling function provided, each vertex weight can be defined as follows.

$$\begin{aligned}
w(z_{t,q}) &= 5s + (t-1)r + q\ ;\ 1 \leq t \leq s,\ 1 \leq q \leq r \\
w(y_t) &= 6s + 2t\ ;\ 1 \leq t \leq s \\
w(x_{1,t}) &= 5s + 3t\ ;\ 1 \leq t \leq s \\
w(x_{2,t}) &= 5s + 3t\ ;\ 1 \leq t \leq s \\
w(z) &= 5rs^2 + 5s^2 + s + \frac{r^2 s(s-1)}{2} + \frac{sr(r+1)}{2}.
\end{aligned}$$

These weight sets will result in a rainbow vertex antimagic coloring for the graph. Next, we will calculate the cardinality of the vertex weight sets.

$W_{2,1} = w(x_{1,t}) = 5s + 3t = \{\, 5s + 3,\ 5s + 6, \ldots,\ 5s + 3s \,\}$

$$\begin{aligned} U_{|W_{2,1}|} &= a + (|W_{2,1}| - 1)\, b \\ 8s &= 5s + 3 + (|W_{2,1}| - 1) \cdot 3 \\ |W_{2,1}| &= s \end{aligned}$$

$$W_{2,2} = w(x_{2,t}) = 5s + 3t = \{\, 5s + 3,\ 5s + 6, \ldots,\ 5s + 3s \,\}$$

$$\begin{aligned} U_{|W_{2,2}|} &= a + (|W_{2,2}| - 1)\, b \\ 8s &= 5s + 3 + (|W_{2,2}| - 1) \cdot 3 \\ |W_{2,2}| &= s \end{aligned}$$

$$W_{2,3} = w(y_t) = 6s + 2t = \{\, 6s + 2,\ 6s + 4,\ \ldots,\ 6s + 2s \,\}$$

$$\begin{aligned} U_{|W_{2,3}|} &= a + (|W_{2,3}| - 1)\, b \\ 8s &= 6s + 2 + (|W_{2,3}| - 1) \cdot 2 \\ |W_{2,3}| &= s \end{aligned}$$

$$W_{2,4} = w(y_{t,q}) = 5s + tr + q - r = \{\, 5s + 1,\ 5s + 2,\ \ldots,\ 5s + sr \,\}$$

$$\begin{aligned} U_{|W_{2,4}|} &= a + (|W_{2,4}| - 1)\, b \\ 5s + sr &= 5s + 1 + (|W_{2,4}| - 1) \cdot 1 \\ |W_{2,4}| &= sr \end{aligned}$$

Based on the cardinalities computed above, we know that $W_{2,1} = W_{2,2}$, for $s \geq 3$ that $W_{2,1} \cup W_{2,2} \cup W_{2,3} \subseteq W_{2,4}$. Thus, since the weights in $W_{2,1}$, $W_{2,2}$, and $W_{2,3}$ overlap with those in $W_{2,4}$, they are redundant and do not need to be counted separately.

$$W_{2,5} = w(z) = 4s^2 + s(s+1) + \frac{r^2 s(s+1)}{2} + \frac{sr(r+1)}{2} + sr(5s - r)$$

$$|W_{2,5}| = 1$$

Let the following notation be defined: $W_{2,1} = \{w(x_{1,t})\}$, $W_{2,2} = \{w(x_{2,t})\}$, $W_{2,3} = \{w(y_t)\}$, $W_{2,4} = \{w(y_{t,q})\}$, $W_{2,5} = \{w(z)\}$.

We now proceed to determine the range of each set as follows.

For $W_{2,1}$, we have $w(x_{1,t}) = 5s + 3t$. The minimum value occurs when $t = 1$ which is, $W_{2,1}^{\min} = 5s + 3 \times 1 = 5s + 3$. The maximum value occurs when $t = s$ which is, $W_{2,1}^{\max} = 5s + 3 \times s = 8s$. Since t ranges from $1 \leq t \leq s$, this set contains all integers from $5s+3$ to $5s+3s$, so $W_{2,1} = [\,5s+3,\ 5s+3s\,] \cap \mathbb{Z}$. Thus, the set $W_{2,1}$ contains all integers from $5s + 3$ to $5s + 3s$, in accordance with the definition and values of t.

For $W_{2,2}$, we have $w(x_{2,t}) = 5s + 3t$. The minimum value occurs when $t = 1$ which is, $W_{2,2}^{\min} = 5s + 3 \times 1 = 5s + 3$. The maximum value occurs when $t = s$ which is, $W_{2,2}^{\max} = 5s + 3 \times s = 8s$. Since t ranges from $1 \leq t \leq s$, this set contains all integers from $5s+3$ to $5s+3s$, so $W_{2,2} = [\,5s+3,\ 5s+3s\,] \cap \mathbb{Z}$. Thus, the set $W_{2,2}$ contains all integers from $5s + 3$ to $5s + 3s$, in accordance with the definition and values of t.

For $W_{2,3}$, we have $w(y_t) = 6s + 2t$. The minimum value occurs when $t = 1$ which is, $W_{2,3}^{\min} = 6s + 2 \times 1 = 6s + 2$. The maximum value occurs when $t = s$ which is, $W_{2,3}^{\max} = 6s + 2 \times s = 6s + 2s = 8s$. Since t ranges from $1 \leq t \leq s$, this set contains all integers from $6s+2$ to $6s+2s$, so $W_{2,3} = [\,6s+2,\ 6s+2s\,] \cap \mathbb{Z}$. Thus, the set $W_{2,3}$ contains all integers from $6s+2$ to $6s+2s$, in accordance with the definition and values of t.

For $W_{2,4}$, we have $w(y_{t,q}) = 5s + tr + q - r$. The minimum value occurs when $t = 1$ and $q = 1$ which is, $W_{2,4}^{\min} = 5s + 1 \times r + 1 - r = 5s + 1$. The maximum value occurs when $t = s$ and $q = r$ which is, $W_{2,4}^{\max} = 5s + s \times r + r - r = 5s + sr$. Since t and q ranges over all pairs $1 \le t \le s$, $1 \le q \le r$, this set contains all integers from $5s+1$ to $5s+sr$, so $W_{2,4} = [\,5s+2,\ 5s+sr\,] \cap \mathbb{Z}$. Thus, the set $W_{2,4}$ contains all integers from $5s+1$ to $5s+sr$, in accordance with the definition and values of t and q.

Finally, for $W_{2,5}$, we have the weight function $w(z) = 4s^2 + s(s+1) + \frac{r^2 s(s+1)}{2} + \frac{sr(r+1)}{2} + sr(5s-r)$. This is a more complex expression compared to the previous ones, and it's not given as a simple range of integers. However, we can express it in terms of its total value rather than a range. Since this is an algebraic involving multiple terms with s and r, we can treat $w(z)$ as a single value (i.e., a constant) representing the weight for the central vertex z. Therefore $W_{2,5} = \{w(z)\} = \{4s^2 + s(s+1) + \frac{r^2 s(s+1)}{2} + \frac{sr(r+1)}{2} + sr(5s-r)\}$. This is a single element set, meaning $|W_{2,5}| = 1$. Thus, the set $W_{2,5}$ contains just one value, which is the weight of the central vertex z. In conclusion, $W_{2,5}$ is a single element set containing the weight of the central vertex, and its cardinality is 1.

Next, we will examine the intersections and subset relations among these sets. First, from $W_{2,1} = [\,5s+3,\ 5s+3s\,] \cap \mathbb{Z}$ and $W_{2,5} = [\,4s^2 + s(s+1) + \frac{r^2 s(s+1)}{2} + \frac{sr(r+1)}{2} + sr(5s-r)\,]$, it follows that $W_{2,1}^{\max} = 5s + 3s < 5s + sr + 1 = W_{2,5}^{\min}$. Therefore, there is no integer that lies in both sets $W_{2,1}$ and $W_{2,5}$, so we conclude that $W_{2,1} \cap W_{2,5} = \varnothing$.

After that, we compare $W_{2,4}$ with the linear sets $W_{2,1}$, $W_{2,2}$, $W_{2,3}$, and $W_{2,5}$. The largest upper bound among these linear sets is given by $\max\{W_{2,1}^{\max}, W_{2,2}^{\max}, W_{2,3}^{\max}, W_{2,5}^{\max}\} = W_{2,5}^{\max} = 5s + 2rs$. The difference between the lower bound of $W_{2,4}$ and $W_{2,5}^{\max}$ is $\big(5rs + 4s + (r+2) + \frac{r(r-1)}{2}\big) - \big(5s + 2rs\big) = 3rs - s + (r+2) + \frac{r^2 - r}{2}$. This expression is clearly positive for all $s \ge 1$ and $r \ge 1$. Therefore, $W_{2,4}^{\min} > 5s + 2sr$. Since all elements of $W_{2,1}$, $W_{2,2}$, $W_{2,3}$, and $W_{2,5}$ are at most $5s + 2sr$, while every element of $W_{2,4}$ is greater than this value, we conclude that $W_{2,4} \cap (W_{2,1} \cup W_{2,2} \cup W_{2,3} \cup W_{2,5}) = \varnothing$.

Now, we examine the relation among $W_{2,2}, W_{2,3}$ and $W_{2,1}$. Since $W_{2,2} = W_{2,3} = \{5s+3, 5s+6, \ldots, 5s+3s\}$, the sets $W_{2,2}$ and $W_{2,3}$ always overlap completely for all s and r, i.e., $W_{2,2} \cap W_{2,3} = W_{2,2} = W_{2,3}$. To determine when an element of $W_{2,2}$ belongs on $W_{2,1}$, take any t with $1 \le t \le s$. We seek t' and q such that $w(x_{1,t} = w(y_{t',q})$. This equation is equivalent to $5s + 3t = 5s + (t'-1)r + q \iff 3t = (t'-1)r + q$. Since $3t$ is a positive integer, we may write $t' = [\,\frac{3t}{r}\,]$, $q = 3t - (t'-1)r$. From the definition of the function, we have $(t'-1)r < 3t \le t'r$, so $1 \le 3t - (t'-1)r = q \le r$. Thus, whenever t' is defined as above, the index q automatically satisfies $1 \le q \le r$. To ensure that $y_{t',q}$ is indeed a vertex of the graph, we also require $1 \le t' \le s$. From $t' = [\,\frac{3t}{r}\,] \le s$, we obtain $t' = \frac{3t}{r} \le s \iff 3t \le sr$. Therefore, for a given t, the overlap $w(x_{1,t}) = w(y_{t',q})$ occurs if and only if $3t \le sr$.

Thus, the intersection $W_{2,1} \cap W_{2,2}$ is non empty if and only if there exists t with $1 \le t \le s$ and $3t \le rs$. This is equivalent to condition $rs \ge 3$, since it suffices to choose $t = 1$ to obtain $3 \le sr$. Hence $W_{2,1} \cap W_{2,2} \ne \varnothing \iff sr \ge 3$.

The same conclusion holds for $W_{2,3}$ because $W_{2,3} = W_{2,2}$. Moreover, $W_{2,2}$ is a subset of $W_{2,1}$ if every value $5s + 3t$ with $1 \le t \le s$ lies in the interval $5s+1, 5s+sr$. This condition is equivalent to $5s + 3s \le 5s + sr \iff 3s \le sr \iff r \ge 3$.

In other words, for $r \geq 3$, we have $W_{2,2} = W_{2,3} \subseteq W_{2,1}$, and the overlap is a full inclusion. In contrast, for the cases $sr \geq 3$ but $r \geq 3$ (i.e., $r = 1$ or $r = 2$ with $sr \geq 3$), the intersection $W_{2,1} \cap W_{2,2}$ contains only part of the elements of $W_{2,2}$.

Regarding the overlap with $W_{2,5}$ from $W_{2,5} = \left[\, 4s^2 + s(s+1) + \frac{r^2s(s+1)}{2} + \frac{sr(r+1)}{2} + sr(5s-r) \,\right]$ and the upper bound of $W_{2,2}$, which is $5s + 3s$, we see that for $r \geq 3, 5s + 32 \leq 5s + rs + 1 \iff 3s \leq sr + 1$, so that $W_{2,2}^{\max} < W_{2,5}^{\min} \iff W_{2,2} \cap W_{2,5} = \varnothing, W_{2,3} \cap W_{2,5} = \varnothing$. For $r = 1$ or $r = 2$, there may exist some values $5s + 3t$ that lie between $5s + sr + 1$ and $5s + 2sr$, so that the intersections $W_{2,2} \cap W_{2,5}$ and $W_{2,3} \cap W_{2,5}$ may be non empty. However, such overlap does not occur for all s and r.

In summary, the intersection and subset structure can be described as follows. The sets $W_{2,2}$ and $W_{2,3}$ are always identical for all s and r, hence $W_{2,2} \cap W_{2,3} = W_{2,2} = W_{2,3}$. The intersection of $W_{2,1}$ with $W_{2,2}$ and $W_{2,3}$ is non-empty if and only if $rs \geq 3$, and for $r \geq 3$, we have $W_{2,2} = W_{2,3} \subseteq W_{2,1}$. The intersection of $W_{2,1}$ with $W_{2,5}$ is always empty, $W_{2,1} \cap W_{2,5} = \varnothing$, and similarly, $W_{2,4}$ does not intersect with $W_{2,1}$, $W_{2,2}$, or $W_{2,3}$. Thus, the only overlap that occurs for all s and r is the full overlap between $W_{2,2}$ and $W_{2,3}$, while all other overlaps depend on the specific values of s and r according to the conditions above.

Finally, the overlap between elements of $W_{2,1}$ and $W_{2,2}$ (or $W_{2,3}$) can be established by equating the weight functions $w(x_{1,t})$ and $w(y_{t,q})$. From the formulas $w(x_{1,t}) = 5s + 3t$ and $w(y_{t,q}) = 5s + (t-1)r + q$, overlap occurs exactly when $3t = (t-1)r + q$. Solving this equation yields $q = 3t - (t-1)r$, which must satisfy $1 \leq q \leq r$. This condition leads to the constraint $t \leq \dfrac{rs}{3}$ for an overlap to occur. Therefore, the overlap between $W_{2,1}$ and $W_{2,2}$ (or $W_{2,3}$) occurs if and only if $t \leq \dfrac{rs}{3}$.

Based on the cardinalities obtained above, we observe that the sets $W_{2,2}$ and $W_{2,3}$, each with cardinality s, are entirely contained within $W_{2,1}$, whose cardinality is rs. This containment represents the points of overlap, where each element of $W_{2,2}$ and $W_{2,3}$ coincides with some element of $W_{2,1}$, the exact positions of these overlaps depend on the values of s and r. The set $W_{2,4}$, also with cardinality s, lies strictly outside the range of $W_{2,1}$, ensuring that no overlap occurs. Similarly, $W_{2,5}$, with cardinality rs, begins at $s(r+4) + s + 1$, a value well above the maximum elements of both $W_{2,1}$ and $W_{2,4}$, making this set disjoint from all preceding sets. Consequently, the only overlap occurs between $W_{2,2}$ and $W_{2,3}$ with $W_{2,1}$.

By summing all distinct induced colors, we obtain $|W_{2,1}| + |W_{2,4}| + |W_{2,5}| = rs + s + rs + 1 = 2rs + s + 1$. Therefore, the construction provides a lower bound of $rvac \geq sr + 1$, and since the induced coloring attains this value, we also have $rvac \leq sr + 1$, leading to the conclusion that $rvac(Amal(CF_r, z, s)) = sr + 1 \quad \text{for all} \quad s \geq 3 \quad \text{and} \quad r \geq 3.$ □

The rainbow path of this graph can be seen in Table 1 and an illustration of the *rvac* of $Amal(CF_r, z, s)$ n in Figure 2.

TABLE 1. The Rainbow Path u–v Graph $Amal(CF_r, z, s)$.

u	v	**Rainbow path u-v**	**Condition**
y_t	y_{t+1}	$y_t,\ x_{1,t},\ z,\ x_{1,t+1},\ y_{t+1}$	$1 \leq t \leq s$
y_t	$x_{2,t+1}$	$y_t,\ x_{1,t},\ z,\ x_{2,t+1}$	$1 \leq t \leq s$
z	y_t	$z,\ x_{2,t},\ y_t$	$1 \leq t \leq s$

Based on Table 1 every u–v path is a rainbow path, so each path in the graph $Amal(CF_r, z, s)$ has distinct colors on its interior vertices. Then, based on Figure

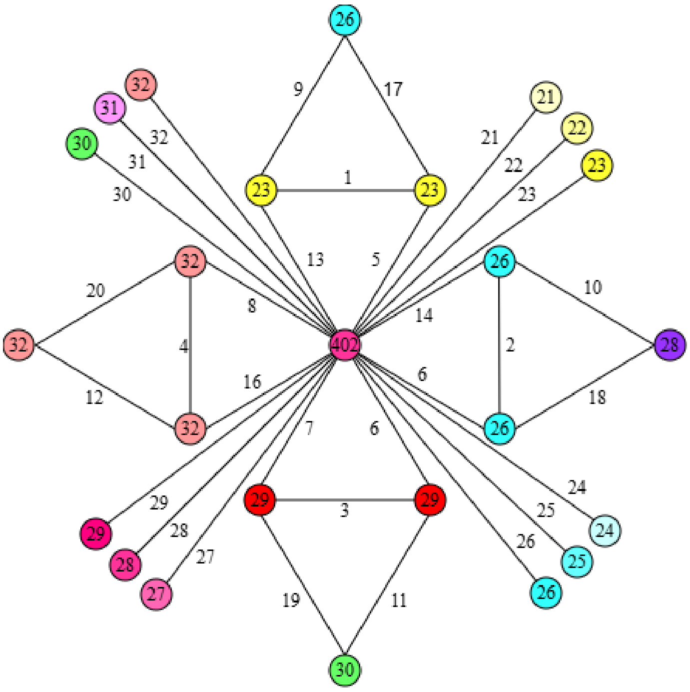

FIGURE 2. $rvac$ of $Amal(CF_3, z, 4)$ graph.

2, the graph $Amal(CF_r, z, s)$ satisfies the rainbow vertex antimagic coloring property. Consequently, the previously obtained lower and upper bounds coincide, and hence $rvac\big(Amal(CF_r, z, s)\big) = sr + 1$, as claimed $rvac(Amal(CF_r, z, s)) = sr + 1$.

Theorem 3. *Let* $Amal(SJ_r, z, s)$ *be an amalgamation of semi jellyfish graph with* $s \geq 3$, $r \geq 3$, *the rainbow vertex connection number of* $Amal(SJ_r, z, s)$ *is* $rvc\big(Amal(SJ_r, z, s)\big) = s + 1$.

Proof. Let $Amal(SJ_r, z, s)$ with vertex set $V(Amal(SJ_r, z, s)) = \{x_{1,t} \mid 1 \leq t \leq s,\} \cup \{x_{2,t} \mid 1 \leq t \leq s,\} \cup \{y_{1,t} \mid 1 \leq t \leq s,\} \cup \ \{y_{2,t} \mid 1 \leq t \leq s,\} \cup \{z_{t,q} \mid 1 \leq t \leq s, 1 \leq q \leq r\} \cup \{z\}$ and edge set $E(Amal(SJ_r, z, s)) = \{x_{1,t}y_{1,t} \mid 1 \leq t \leq s,\} \cup \{x_{2,t}y_{2,t} \mid 1 \leq t \leq s,\} \cup \{y_{1,t}y_{2,t} \mid 1 \leq t \leq s\} \cup \{x_{1,t}x_{2,t} \mid 1 \leq t \leq s\} \cup \{x_{1,t}z \mid 1 \leq t \leq s\} \cup \{x_{2,t}z \mid 1 \leq t \leq s\}\{z_{t,q}z \mid 1 \leq t \leq s, 1 \leq \ q \leq r\}$. Based on these vertex and edge sets, the cardinalities are $|V(Amal(SJ_r, z, s))| = 4s + sr + 1$ and $|E(Amal(SJ_r, z, s))| = 6s + sr$.

Next, we will prove that $rvc(Amal(SJ_r, z, s)) = s + 1$ by showing the lower bound and upper bounds. First, we will prove the lower bound. To prove the lower bound, we use to Lemma 1, which states the $rvc(G) \geq diam(G) - 1$. Since $(Amal(SJ_r, z, s))$ is a connected graph, and the diameter of the graph $diam(Amal(SJ_r, z, s))$ is 4, by Lemma 1 we can conclude $rvc(Amal(SJ_r, z, s)) \geq diam(Amal(SJ_r, z, s)) - 1 = 4 - 1 = 3$. This gives a general lower bound $rvc(Amal(SJ_r, z, s)) \geq 3$.

However, we can obtain a stronger lower bound by further analyzing the graphs structure and the longest paths in it. Lets consider pairs of vertices $y_{1,t}$ and $y_{1,t+1}$ for $1 \geq t \geq s$. Each path connecting such a pair must pass through several internal vertices, which forms the longest path as follows $y_{1,t} \rightarrow x_{1,t} \rightarrow z \rightarrow x_{1,t+1} \rightarrow y_{1,t+1}$. Since this path must be a rainbow path (each internal vertex must have a different color), we need at least $s + 1$ distinct colors to ensure that the paths are correctly connected.

By considering all adjacent branches for each $t = 1, 2, \ldots, s$, we observe that the vertices $y_{1,1}, y_{1,2}, \ldots, y_{1,s}$, the vertices $y_{2,1}, y_{2,2}, \ldots, y_{2,s}$, the vertices $x_{1,1}, x_{1,2}, \ldots, x_{1,s}$, the vertices $x_{2,1}.x_{2,2}, \ldots, x_{2,s}$, and the central vertex z must all receive different colors. Therefore, at least $s + 1$ colors are required to color this graph. Thus, we obtain the stronger lower bound is $rvc(Amal(SJ_r, z, s)) \geq s + 1$.

Second, to prove the upper bound of $rvc(Amal(SJ_r, z, s)) \leq s+1$ by defining the vertex function as follows.

$$f(y_{1,t}) = f(y_{2,t}) = f(z_{t,q}) = f(z) = 1\ ;\ 1 \leq t \leq s,\ 1 \leq q \leq r,$$
$$f(x_{1,t}) = f(x_{2,t}) = t + 1\ ;\ 1 \leq t \leq s$$

We assume C_1 as the color set on $y_{1,t}, y_{2,t}z_{t,q}$, and z so that $C_1 = \{1\} \Longrightarrow |C_1| = 1$. Next, we assume C_2 as the color set of the vertices $x_{1,t}$ and $x_{2,t}$ so that $C_2 = \{2, 3, \cdots, s+1\} \Longrightarrow |C_2| = s$. Based on this, the total color cardinality is $|C_1| + |C_2| = 1 + s = s + 1$. Then, we have $rvc(Amal(SJ_r, z, s)) \leq s + 1$. Based on the lower and the upper bound, we have $s + 1 \leq rvc(Amal(SJ_r, z, s)) \leq s + 1$. Then, it concludes that $rvc(Amal(SJ_r, z, s)) = s + 1$. □

The rainbow path of the graph $Amal(SJ_r, z, s)$ can be seen in Table 2 and an illustration of the rainbow vertex coloring of $Amal(SJ_r, z, s)$ see in Figure 3.

Based on Table 2 every u–v path is a rainbow path, so each path in the graph $Amal(SJ_r, z, s)$ has distinct colors on its interior vertices. Then, based on Figure3, the graph $Amal(SJ_r, z, s)$ satisfies the rainbow vertex connection property. Consequently, the previously obtained lower and upper bounds coincide, and hence $rvc\big(Amal(SJ_r, z, s)\big) = s + 1$, as claimed $rvc(Amal(SJ_r, z, s)) = s + 1$.

Theorem 4. *Let* $Amal(SJ_r, z, s)$ *be an amalgamation semi jellyfish graph with* $s \geq 3, r \geq 3$, *the rainbow vertex antimagic connection number of* $Amal(SJ_r, z, s)$ *is* $rvac$ $(Amal(SJ_r, z, s)) = sr + 1$.

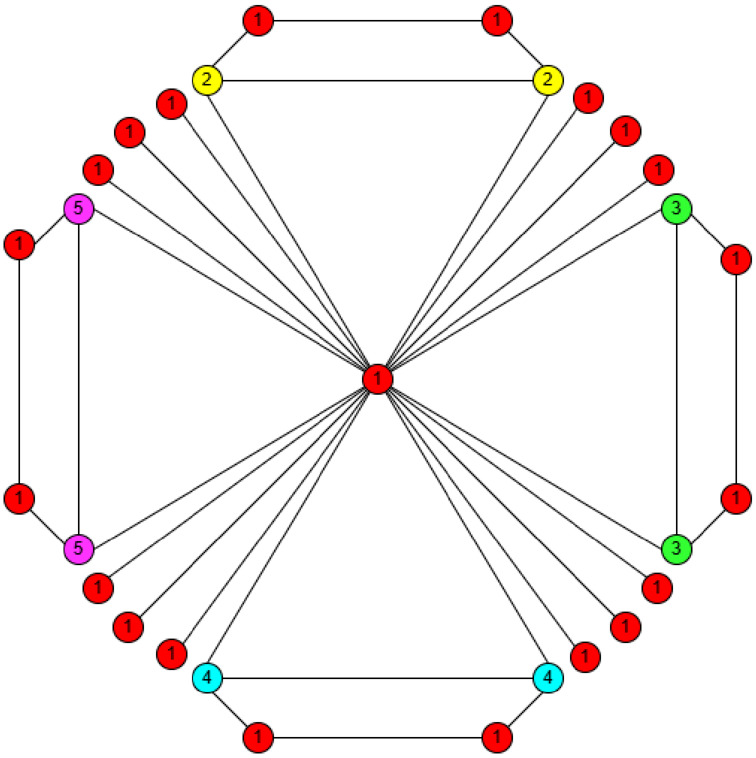

FIGURE 3. rvc of graph $Amal(SJ_3, z, 4)$.

Proof. Let $Amal(SJ_r, z, s)$ with the vertex set $V(Amal(SJ_r, z, s)) = \{x_{1,t} \mid 1 \leq t \leq s,\} \cup \{x_{2,t} \mid 1 \leq t \leq s,\} \cup \{y_{1,t} \mid 1 \leq t \leq s,\} \cup \{y_{2,t} \mid 1 \leq t \leq s,\} \cup \{z_{t,q} \mid 1 \leq t \leq s, 1 \leq q \leq r\} \cup \{z\}$ and edge set $E(Amal(SJ_r, z, s)) = \{x_{1,t}y_{1,t} \mid 1 \leq t \leq s,\} \cup \{x_{2,t}y_{2,t} \mid 1 \leq t \leq s,\} \cup \{y_{1,t}y_{2,t} \mid 1 \leq t \leq s\} \cup \{x_{1,t}x_{2,t} \mid 1 \leq t \leq s\} \cup \{x_{1,t}z \mid 1 \leq t \leq s\} \cup \{x_{2,t}z \mid 1 \leq t \leq s\}\{z_{t,q}z \mid 1 \leq t \leq s, 1 \leq \ q \leq r\}$. Based on these vertex and edge sets, the cardinalities are $|V(Amal(SJ_r, z, s))| = 4s + sr + 1$ and $|E(Amal(SJ_r, z, s))| = 6s + sr$.

Next, we will prove that $rvac(Amal(SJ_r, z, s)) = sr + 1$ by showing the lower bound and upper bounds. First, we will prove the lower bound. To prove the lower bound, we use Lemma 3, which states that for a connected graph G, we have $rvac(G) \geq max\{rvc(G), |P(G)|\}$, where $rvc(G)$ is the rainbow vertex connection number of graph G, and $P(G)$ is the set of pendant vertices of G. For the graph $Amal(SJ_r, z, s)$, we know that the rainbow vertex connection number $rvc(Amal(SJ_r, z, s)) = s + 1$ from the previous results (Theorem 3). Thus, we have $rvac(Amal(SJ_r, z, s)) \geq max\{s + 1, |P(Amal(SJ_r, z, s))|\}$.

We calculate $|P(Amal(SJ_r, z, s))|$. The graph $Amal(SJ_r, z, s)$ contains sr pendant vertices, specifically the vertices $z_{t,q}$ for $1 \leq t \leq s$ and $1 \leq q \leq r$, each of which is only connected to the central vertex z. Therefore, the number of pendant vertices $|P(Amal(SJ_r, z, s))$ is sr. Thus, we obtain $rvac(Amal(SJ_r, z, s)) \geq max\{s+1, sr\}$. Since $sr \geq s+1$ for all $s \geq 3$ and $r \geq 3$ we can conclude that $rvac(Amal(SJ_r, z, s)) \geq sr$. Since $|P(G)| \geq rvc(G)$, we use the inequality for $rvac(G) \geq |P(G)|$ on this graph according to Lemma 3.

However, to ensure the rainbow vertex antimagic coloring on this graph, we need to assign to additional color for the central vertex z, contributing one extra color. Therefore, the total number of colors required for the coloring is at least $sr+1$, consisting of sr colors for the pendant vertices and one color for the central vertex. Thus, we obtain the lower bound for $rvac(Amal(SJ_r, z, s))$ is $rvac(Amal(SJ_r, z, s)) \geq sr + 1$.

Second, prove the upper bound of $rvac(Amal(SJ_r, z, s)) \leq rs + 1$ by defining the vertex function as follows.

$$\begin{aligned}
f(x_{1,t}\,x_{2,t}) &= t\ ;\ 1 \leq t \leq s\\
f(x_{2,t}\,y_{2,t}) &= 2s - t + 1\ ;\ 1 \leq t \leq s\\
f(x_{1,t}\,y_{1,t}) &= 3s - t + 1\ ;\ 1 \leq t \leq s\\
f(x_{1,t}\,z) &= 3s + t\ ;\ 1 \leq t \leq s\\
f(x_{2,t}\,z) &= 4s + t\ ;\ 1 \leq t \leq s\\
f(y_{1,t}\,y_{2,t}) &= 5s + t\ ;\ 1 \leq t \leq s\\
f(z\,z_{t,q}) &= 6s + tr + q - r\ ;\ 1 \leq t \leq s,\ 1 \leq q \leq r
\end{aligned}$$

According to the labeling function provided, each vertex weight can be defined as follows.

$$\begin{aligned}
w(x_{1,t}) &= 6s + t + 1\ ;\ 1 \leq t \leq s,\ 1 \leq q \leq r\\
w(x_{2,t}) &= 6s + t + 1\ ;\ 1 \leq t \leq s,\ 1 \leq q \leq r\\
w(y_{2,t}) &= 7s + 1\ ;\ 1 \leq t \leq s\\
w(y_{1,t}) &= 8s + 1\ ;\ 1 \leq t \leq s\\
w(z_{t,q}) &= 6s + tr + q - r\ ;\ 1 \leq t \leq s, 1 \leq q \leq r\\
w(z) &= 8s^2 + s + 6s^2r + \frac{s^2r^2 + sr}{2}.
\end{aligned}$$

These weight sets will result in a rainbow vertex antimagic coloring for the graph. Next, we will calculate the cardinality of the vertex weight sets.

$$W_{2,1} = w(x_{1,t}) = 6s + t + 1 = \{\,6s + 2,\ 6s + 3, \ldots,\ 6s + s + 1\,\}$$

$$\begin{aligned}
U_{|W_{2,1}|} &= a + (|W_{2,1}| - 1)\,b\\
7s + 1 &= 6s + 2 + (|W_{2,1}| - 1) \cdot 1\\
|W_{2,1}| &= s
\end{aligned}$$

$$W_{2,2} = w(x_{2,t}) = 6s + t + 1 = \{\,6s + 2,\ 6s + 3, \ldots,\ 6s + s + 1\,\}$$

$$\begin{aligned}
U_{|W_{2,2}|} &= a + (|W_{2,2}| - 1)\,b\\
7s + 1 &= 6s + 2 + (|W_{2,2}| - 1) \cdot 1\\
|W_{2,2}| &= s
\end{aligned}$$

$$W_{2,3} = w(y_{1,t}) = 8s + 1$$

$$\begin{aligned}
U_{|W_{2,3}|} &= a + (|W_{2,3}| - 1)\,b\\
8s + 1 &= 8s + 1 + (|W_{2,3}| - 1) \cdot 0\\
|W_{2,3}| &= 1
\end{aligned}$$

$$W_{2,4} = w(y_{2,t}) = 7s + 1$$

$$\begin{aligned} U_{|W_{2,4}|} &= a + (|W_{2,4}| - 1)\, b \\ 7s + 1 &= 7s + 1 + (|W_{2,4}| - 1) \cdot 0 \\ |W_{2,4}| &= 1 \end{aligned}$$

$$W_{2,5} = w(z_{t,q}) = 6s + tr + q - r = \{\, 6s + 1,\, 6s + 2, \ldots,\, 6s + sr \,\}$$

$$\begin{aligned} U_{|W_{2,5}|} &= a + (|W_{2,5}| - 1)\, b \\ 6s + sr &= 6s + 1 + (|W_{2,5}| - 1) \cdot 1 \\ |W_{2,5}| &= sr \end{aligned}$$

$$W_{2,6} = w(z) = \left\{ 8s^2 + s + 6s^2 r + \frac{s^2 r^2 + sr}{2} \right\}$$

$$\begin{aligned} U_{|W_{2,6}|} &= a + (|W_{2,6}| - 1)\, b \\ |W_{2,6}| &= 1 \end{aligned}$$

Based on the cardinalities computed above, we have $W_{2,1} = W_{2,2}$, so these two sets represent the same collection of vertex weights and should be counted only once. Furthermore, observe that $W_{2,1} = \{6s + 2, 6s + 3, \ldots, 6s + s + 1\}$ is contained in $W_{2,5} = \{6s + 1, 6s + 2, \ldots, 6s + sr\}$, since the smallest element of $W_{2,1}$ is at least $6s + 1$ and the largest element of $W_{2,1}$ is at most $6s + sr$. Next, the set $W_{2,4} = \{7s + 1\}$, which consists of a single element, is also contained in $W_{2,5}$ whenever $7s + 1 \leq 6s + sr$, that is, whenever $s + 1 \leq sr$. Similarly, the set $W_{2,3} = \{8s + 1\}$, which also consists of a single element, is contained in $W_{2,5}$ whenever $8s + 1 \leq 6s + sr$, namely whenever $2s + 1 \leq sr$. Thus, for $r \geq 3$, both $W_{2,3}$ and $W_{2,4}$ are already contained in $W_{2,5}$. Hence, in determining the number of distinct vertex weights, it is sufficient to consider only $W_{2,5}$ and $W_{2,6}$, because $W_{2,1}$ and $W_{2,2}$ are identical and both are contained in $W_{2,5}$, while $W_{2,3}$ and $W_{2,4}$ are likewise contained in $W_{2,5}$.

Let the following notation be defined: $W_{2,1} = \{w(x_{1,t})\}$, $W_{2,2} = \{w(x_{2,t})\}$, $W_{2,3} = \{w(y_{1,t})\}$, $W_{2,4} = \{w(y_{2,t})\}$, $W_{2,5} = \{w(z_{t,q})\}$, $W_{2,6} = \{w(z)\}$.

We now proceed to determine the range of each set of induced vertex weights.

For $W_{2,1}$, recall that $w(x_{1,t}) = 6s + t + 1$ for $1 \leq t \leq s$. Since t starts from 1, the minimum value of $w(x_{1,t})$ is obtained when $t = 1$, namely $W_{2,1}^{\min} = 6s + 2$. On the other hand, since $t \leq s$, the maximum value is attained when $t = s$, namely $W_{2,1}^{\max} = 6s + s + 1$. Hence, as t varies from 1 to s, the values of $w(x_{1,t})$ form an arithmetic progression with common difference 1. Therefore, $W_{2,1} = \{\, 6s + 2,\, 6s + 3, \ldots,\, 6s + s + 1 \,\}$.

Similarly, for $W_{2,2}$, we have $w(x_{2,t}) = 6s + t + 1$ for $1 \leq t \leq s$. By the same argument, the minimum value is $W_{2,2}^{\min} = 6s + 2$, and the maximum value is $W_{2,2}^{\max} = 6s + s + 1$. Thus, the set $W_{2,2}$ is also an arithmetic progression with common difference 1, namely $W_{2,2} = \{\, 6s + 2,\, 6s + 3, \ldots,\, 6s + s + 1 \,\}$. Hence, we immediately obtain $W_{2,1} = W_{2,2}$.

Next, for $W_{2,3}$, we have $w(y_{1,t}) = 8s + 1$. Since this expression does not depend on t, every vertex $y_{1,t}$ has the same induced weight. Therefore, $W_{2,3} = \{\, 8s + 1 \,\}$, so $|W_{2,3}| = 1$.

For $W_{2,4}$, we have $w(y_{2,t}) = 7s + 1$. Again, this value is independent of t, so every vertex $y_{2,t}$ has the same induced weight. Hence, $W_{2,4} = \{\, 7s + 1 \,\}$, and consequently $|W_{2,4}| = 1$.

Now consider $W_{2,5}$. By definition, $w(z_{t,q}) = 6s + tr + q - r$, where $1 \leq t \leq s$ and $1 \leq q \leq r$. To determine the smallest value in this set, we take the smallest possible values of t and q, namely $t = 1$ and $q = 1$. This gives $W_{2,5}^{\min} = 6s+r+1-r = 6s+1$. To determine the largest value, we take $t = s$ and $q = r$, which yields $W_{2,5}^{\max} = 6s+sr+r-r = 6s+sr$. Moreover, as (t, q) ranges over all possible pairs, the expression $tr+q-r$ runs consecutively through all integers from 1 up to sr. Therefore, $W_{2,5} = \{\, 6s+1,\, 6s+2, \ldots,\, 6s+sr \,\}$.

Finally, for $W_{2,6}$, we have $w(z) = 8s^2 + s + 6s^2r + \frac{s^2r^2+sr}{2}$. Since this is the weight of a single central vertex, it follows that $W_{2,6} = \left\{\, 8s^2 + s + 6s^2r + \frac{s^2r^2+sr}{2} \,\right\}$, and hence $|W_{2,6}| = 1$.

Next, we examine the intersections among these sets in order to determine the number of distinct induced colors.

First, since $W_{2,1} = \{\, 6s+2,\, 6s+3, \ldots,\, 6s+s+1 \,\}$ and $W_{2,2} = \{\, 6s+2,\, 6s+3, \ldots,\, 6s+s+1 \,\}$, it follows directly that $W_{2,1} = W_{2,2}$. Thus, these two sets coincide completely, and so they contribute the same induced colors. Therefore, they must be counted only once in determining the total number of distinct colors.

Next, we compare $W_{2,1}$ and $W_{2,2}$ with $W_{2,5}$. Since $W_{2,5} = \{\, 6s+1,\, 6s+2, \ldots,\, 6s+sr \,\}$, every element of $W_{2,1}$ and $W_{2,2}$ belongs to $W_{2,5}$. Hence, $W_{2,1} \subseteq W_{2,5}$ and $W_{2,2} \subseteq W_{2,5}$.

Now consider $W_{2,4} = \{\, 7s+1 \,\}$. This element belongs to $W_{2,5}$ whenever $7s+1 \leq 6s+sr$, which is equivalent to $s+1 \leq sr$. Since $s \geq 3$ and $r \geq 3$, this inequality holds. Therefore, $W_{2,4} \subseteq W_{2,5}$.

Similarly, for $W_{2,3} = \{\, 8s+1 \,\}$, we have $8s+1 \leq 6s+sr$ if and only if $2s+1 \leq sr$. Again, for $s \geq 3$ and $r \geq 3$, this condition is satisfied. Hence, $W_{2,3} \subseteq W_{2,5}$.

Therefore, all induced vertex weights corresponding to the vertices $x_{1,t}$, $x_{2,t}$, $y_{1,t}$, $y_{2,t}$, and $z_{t,q}$ are already contained in $W_{2,5}$. The only induced vertex weight not included in this set is the weight of the central vertex z, represented by $W_{2,6}$.

Consequently, the set of all distinct induced vertex weights is given by $\bigcup_{i=1}^{6} W_{2,i} = W_{2,5} \cup W_{2,6}$. Since $W_{2,5}$ contains sr elements and $W_{2,6}$ contains exactly one element, we obtain

$$\left| \bigcup_{i=1}^{6} W_{2,i} \right| = |W_{2,5} \cup W_{2,6}| = |W_{2,5}| + |W_{2,6}| = sr + 1.$$

Therefore, the number of distinct induced colors is $sr + 1$.

Thus, we obtain $rvac(Amal(SJ_r, z, s)) \geq sr + 1$. On the other hand, the labeling defined above yields an induced rainbow vertex antimagic coloring using exactly $sr + 1$ colors. Hence, $rvac(Amal(SJ_r, z, s)) \leq sr + 1$. Combining both inequalities, we conclude that $rvac(Amal(SJ_r, z, s)) = sr + 1$, for $s \geq 3$ and $r \geq 3$.

□

The rainbow path of this graph can be seen in Table 2 and an illustration of the *rvac* of $Amal(SJ_r, z, s)$ can be seen in Figure 4.

TABLE 2. Rainbow path $Amal(SJ_r, z, s)$

u	v	**Rainbow path** $u - v$	**Condition**
y_t	y_{t+1}	$y_t,\ x_{1,t},\ z,\ x_{1,t+1},\ y_{t+1}$	$1 \leq t \leq s$
y_t	$x_{2,t+1}$	$y_t,\ x_{1,t},\ z,\ x_{2,t+1}$	$1 \leq t \leq s$
z	y_t	$z,\ x_{2,t},\ y_t$	$1 \leq t \leq s$

Based on Table 2 every u–v path is a rainbow path, so each path in the graph $Amal(SJ_r, z, s)$ has distinct colors on its interior vertices. Then, based on Figure

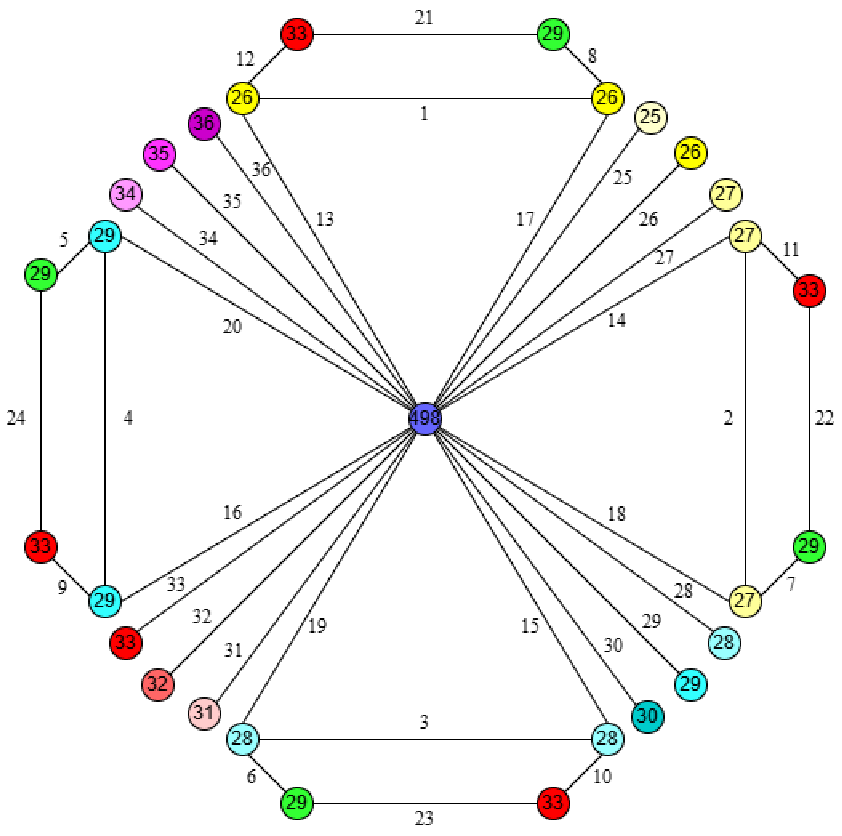

Figure 4. $rvac$ of $Amal(SJ_3, z, 4)$ graph.

4, the graph $Amal(SJ_r, z, s)$ satisfies the rainbow vertex antimagic coloring property. Consequently, the previously obtained lower and upper bounds coincide, and hence $rvac\big(Amal(SJ_r, z, s)\big) = sr + 1$, as claimed $rvac(Amal(SJ_r, z, s)) = sr + 1$.

Theorem 5. *Let $Amal(D_r, z, s)$ be an amalgamation of diamond graph with $s \geq 2, r \geq 2$, the rainbow vertex connection number of $Amal(D_r, z, s)$ is $rvc\big(Amal(D_r, z, s)\big) = 3$.*

Proof. Let $Amal(D_r, z, s)$ with vertex set $V(G) = \{x_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{y_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{z_t \mid 1 \leq t \leq s\} \cup \{a\}$ and edge set $E(G) = \{zx_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{zy_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{x_{t,q}, Z_t \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{y_{t,q}z_t \mid 1 \leq t \leq s,\ 1 \leq q \leq r\}$. Based on these vertex and edge sets, the cardinalities are $|V(Amal(D_r, z, s))| = 2sr + s + 1$ and $|E(Amal(D_r, z, s))| = 4sr$.

Next, we will prove that $Amal(D_r, z, s)$ by showing the lower bound and upper bounds. First, we will prove the lower bound. To prove the lower bound, we use Lemma 1, which states the $rvc(G) \geq diam(G) - 1$. Since $Amal(D_r, z, s)$ is a connected graph, and the diameter of the graph $diam(Amal(D_r, z, s))$ is 4, by Lemma 1 we can conclude $rvc(Amal(D_r, z, s)) \geq diam(Amal(D_r, z, s)) - 1 = 4 - 1 = 3$. This gives a general lower bound $rvc(Amal(D_r, z, s)) \geq 3$.

Take a path between $z_t z_{t+1}$ then $z_t \neq z_{t+1}$ since all internal vertices along the path must receive distinct colors. For example, the edge $z_1 z_2$ admits the rainbow path $\{y_{1,1}; z; y_{2,2}\}$. Therefore, we obtain the lower bound $rvc(Amal(D_r, z, s)) \geq 3$.

Second, prove the upper bound of $rvc(Amal(D_r, z, s)) \leq 3$ by defining the vertex function as follows.

$$f(z_t) = f(z) = 1;\ 1 \leq t \leq s$$
$$f(x_{t,q}) = f(y_{t,q}) = 2\ ;\ 1 \leq t \leq s, 1 \leq q \leq r$$
$$f(x_{t,q}) = f(y_{t,q}) = 3;\ 1 \leq t \leq s, 1 \leq q \leq r.$$

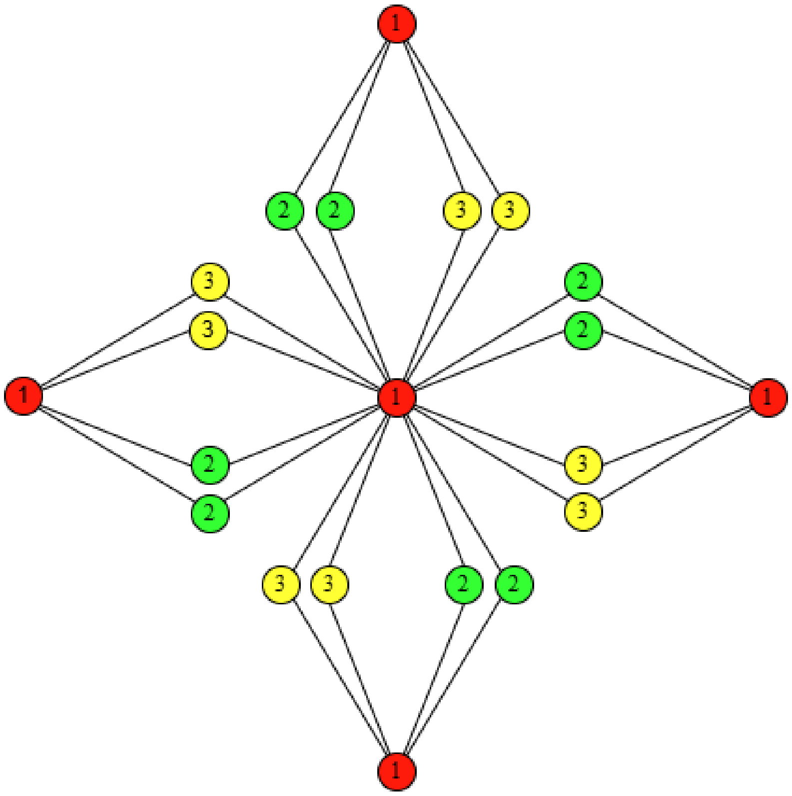

Figure 5. RVC of graph $Amal(D_2, z, 4)$.

We assume C_1 as the color set on z_t and z so that $C_1 = \{1\} \implies |C_1| = 1$. Next, we assume C_2 as the color set of the vertices $x_{1,t}$ and $y_{t,q}$so that $C_2 = \{2\} \implies |C_2| = 1$. Finally, assume C_3 as the color set of the vertices $x_{t,q}$ and $y_{t,q}$ so that $C_3 = \{3\} \implies |C_3| = 1$. Based on this, the total color cardinality is $|C_1| + |C_2| + |C_3| = 1 + 1 + 1 = 3$. Then, we have $rvc(Amal(D_r, z, s)) \leq 3$. Based on the lower and the upper bound, we have $3 \leq rvc(Amal(D_r, z, s)) \leq 3$. Then, it concludes that $rvc(Amal(D_r, z, s)) = 3$. □

The rainbow path of the graph $Amal(D_r, z, s)$ can be seen in Table 3 and an illustration of the rainbow vertex coloring of $Amal(D_r, z, s)$ can be seen in Figure 5.

Based on Table 3 every u–v path admits a rainbow path; in other words, each path in the graph $Amal(D_r, z, s)$ has pairwise distinct colors on its interior vertices. Then, based on Figure 5, the graph $Amal(D_r, z, s)$ satisfies the rainbow vertex connection property. Consequently, the previously derived lower and upper bounds coincide, and hence $rvc\big(Amal(D_r, z, s)\big) = 3$, as claimed $rvc(Amal(D_r, z, s)) = 3$.

Theorem 6. *Let $Amal(D_r, z, s)$ be an amalgamation of diamond graph with $s \geq 2, r \geq 2$, the rainbow vertex antimagic connection number of $Amal(D_r, z, s)$ is $rvac(Amal(D_r, z, s)) = 4$.*

Proof. Let $Amal(D_r, z, s)$ with vertex set $V(G) = \{x_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{y_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{z_t \mid 1 \leq t \leq s\} \cup \{z\}$ and edge set $E(G) = \{zx_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{zy_{t,q} \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{x_{t,q}, z_t \mid 1 \leq t \leq s,\ 1 \leq q \leq r\} \cup \{y_{t,q}z_t \mid 1 \leq t \leq s,\ 1 \leq q \leq r\}$. Based on these vertex and edge sets, the cardinalities are $|V(G)| = 2sr + s + 1$ and $|E(G)| = 4sr$, respectively.

Based on Lemma 1 and Lemma 2, we determine the lower bound of $rvc(Amal(D_r, z, s))$. We have that $rvc(Amal(D_r, z, s)) = 3$, it folllows $rvc(Amal(D_r, z, s)) \geq rvc(Amal(D_r, z, s)) = 3$. However, we cannot have $rvac(Amal(D_r, z, s)) \geq 3$, instead, it follows that $rvac(Amal(D_r, z, s)) \geq 4$. This is because any path from one outer vertex to another, passing through the intermediate vertices and the central vertex, requires distinct colors. Therefore, the total number of colors needed for the graph $Amal(D_r, z, s)$ is at least four: one color for the outer vertices and three different colors for the internal vertices, including the central vertex. It concludes the lowerbound is $rvac(Amal(D_r, z, s)) \geq 4$.

Second, we establish the upper bound for the $rvac$ on the graph $Amal(D_r, z, s)$ namely $rvac(Amal(D_r, z, s)) \leq 4$. We define the label function as follows.

$$\begin{aligned}
f(x_{t,q}\, z) &= t + qs - s\ ;\ 1 \leq t \leq s, 1 \leq q \leq r\\
f(y_{t,q}\, z_t) &= mn + t + qs - s\ ;\ 1 \leq t \leq s, 1 \leq q \leq r\\
f(x_{t,q}\, z_t) &= 3rs - t + 1 + s - qs\ ;\ 1 \leq t \leq s, 1 \leq q \leq r\\
f(y_{t,q}\, z) &= 4rs + s - qs - t + 1\ ;\ 1 \leq t \leq s, 1 \leq q \leq r
\end{aligned}$$

According to the labeling function provided, each vertex weight can be defined as follows.

$$\begin{aligned}
w(x_{t,q}) &= 3rs + 1; 1 \leq t \leq s, 1 \leq q \leq r\\
w(y_{t,q}) &= 5rs + 1; 1 \leq t \leq s, 1 \leq q \leq r\\
w(z_i) &= 4r^2s + r; 1 \leq i \leq s, 1 \leq q \leq r\\
w(z) &= 4r^2s^2 + rs; 1 \leq t \leq s, 1 \leq q \leq r
\end{aligned}$$

Next, we proceed to determine the cardinality of the vertex weight sets in order to obtain the variation number of the induced weights when considered as vertex colors. Suppose that $W_{7,1} = w(y_{t,q}), W_{7,2} = w(x_{t,q}), W_{7,3} = w(z_i), W_{7,4} = w(z)$. We have the followings:

$W_{7,1} = w(y_{t,q}) = 5rs + 1$

$$\begin{aligned}
U_{|W_{7,1}|} &= a + (|W_{7,1}| - 1)\, b,\\
U_{|W_{7,1}|} &= (5rs + 1) + (|W_{7,1}| - 1) \cdot 0,\\
|W_{7,1}| &= 1.
\end{aligned}$$

$W_{7,2} = w(x_{t,q}) = 3rs + 1$

$$\begin{aligned}
U_{|W_{7,2}|} &= a + (|W_{7,2}| - 1)\, b,\\
U_{|W_{7,2}|} &= (3rs + 1) + (|W_{7,2}| - 1) \cdot 0,\\
|W_{7,2}| &= 1.
\end{aligned}$$

$W_{7,3} = w(z_t) = 4r^2s + r$

$$\begin{aligned} U_{|W_{7,3}|} &= a + (|W_{7,3}| - 1)\, b, \\ U_{|W_{7,3}|} &= 4r^2s + r + (|W_{7,3}| - 1) \cdot 0, \\ |W_{7,3}| &= 1. \end{aligned}$$

$W_{7,4} = w(z) = 4r^2s^2 + rs$

$$\begin{aligned} U_{|W_{7,4}|} &= a + (|W_{7,4}| - 1)\, b, \\ U_{|W_{7,4}|} &= 4r^2s^2 + rs + (|W_{7,4}| - 1) \cdot 0, \\ |W_{7,4}| &= 1. \end{aligned}$$

Now we examine the intersections of the weight sets and prove that some of these sets are pairwise disjoint. The set $W_{7,1}$; $w(y_{t,q}) = 5rs + 1 \quad \text{for } 1 \leq t \leq s,\ 1 \leq q \leq r$. Minimum ($t = 1$); $W_{7,1}^{\min} = 5rs + 1$. Maximum ($t = s$); $W_{7,1}^{\max} = 5rs + 1$. Thus, $W_{7,1} \subseteq \{5rs + 1\}$. The set $W_{7,2}$; $w(x_{t,q}) = 3rs + 1 \quad \text{for } 1 \leq t \leq s,\ 1 \leq q \leq r$. Minimum ($t = 1$): $W_{7,2}^{\min} = 3rs + 1$. Maximum ($t = s$); $W_{7,2}^{\max} = 3rs + 1$. Thus, $W_{7,2} \subseteq \{3rs + 1\}$. The set $W_{7,3}$; $w(z_t) = 4r^2s + r \quad \text{for } 1 \leq t \leq s,\ 1 \leq q \leq r$. Minimum ($t = 1$); $W_{7,3}^{\min} = 4r^2s + r$. Maximum ($t = s$); $W_{7,3}^{\max} = 4r^2s + r$. Thus, $W_{7,3} \subseteq \{4r^2s + r\}$. The set $W_{7,4}$; $w(z) = 4r^2s^2 + rs \quad \text{for } 1 \leq t \leq s,\ 1 \leq q \leq r$. Minimum ($t = 1$); $W_{7,4}^{\min} = 4r^2s^2 + rs$. Maximum ($t = s$); $W_{7,4}^{\max} = 4r^2s^2 + rs$. Thus, $W_{7,4} \subseteq \{4r^2s^2 + rs\}$. Now we prove that the intersections of these weight sets are empty, namely: $W_{7,1} \cap W_{7,2} = \varnothing$. The value of $W_{7,1}$ is $5rs + 1$, and the value of $W_{7,2}$ is $3rs + 1$. Since $5rs + 1 > 3rs + 1$ for all $s, r \geq 1$, the ranges of $W_{7,1}$ and $W_{7,2}$ do not overlap. Hence, $W_{7,1} \cap W_{7,2} = \varnothing$.

Next, we prove the remaining intersections. For $W_{7,1}$ and $W_{7,3}$; $5rs + 1 \neq 4r^2s + r$. Since these expressions are never equal for any $s, r \geq 1$, we obtain $W_{7,1} \cap W_{7,3} = \varnothing$. For $W_{7,1}$ and $W_{7,4}$; $5rs + 1 \neq 4r^2s^2 + rs$. Because these two quantities lie on very different scales, no element of $W_{7,1}$ belongs to $W_{7,4}$. Therefore, $W_{7,1} \cap W_{7,4} = \varnothing$. For $W_{7,2}$ and $W_{7,3}$; $3rs + 1 \neq 4r^2s + r$. Since $3rs + 1$ and $4r^2s + r$ also lie on different scales, we obtain $W_{7,2} \cap W_{7,3} = \varnothing$. For $W_{7,2}$ and $W_{7,4}$; $3rs + 1 \neq 4r^2s^2 + rs$. Hence, $W_{7,2} \cap W_{7,4} = \varnothing$. For $W_{7,3}$ and $W_{7,4}$; $4r^2s + r \neq 4r^2s^2 + rs$. Since these expressions again lie on different scales, we deduce that $W_{7,3} \cap W_{7,4} = \varnothing$. For $W_{7,5}$, we have $W_{7,5} = \{4r^2s^2 + rs\}$. Since the value $4r^2s^2 + rs$ is greater than the maximum values of $W_{7,1}$, $W_{7,2}$, $W_{7,3}$, and $W_{7,4}$, we conclude that $W_{7,5} \cap (W_{7,1} \cup W_{7,2} \cup W_{7,3} \cup W_{7,4}) = \varnothing$. From the above analysis, we conclude that the intersection of all weight sets is empty, namely $W_{7,1} \cap W_{7,2} \cap W_{7,3} \cap W_{7,4} \cap W_{7,5} = \varnothing$.

Since each weight set consists of a fixed value that is clearly distinct and mutually non-overlapping, except possibly for some specific values between $W_{7,1}$ and $W_{7,2}$ that may share the same weight, we may conclude that no set is contained in any other set. Each weight set is therefore a separate set, except for those particular values in $W_{7,1}$ and $W_{7,2}$ that may coincide, even though their defining mathematical expressions are not identical.

The computation of vertex weights in the graph $Amal(D_r, z, s)$ shows that all resulting vertex-weight sets are mutually distinct and each has a constant cardinality of one. The four weight sets obtained, namely $W_{7,1} = 5rs + 1$, $W_{7,2} = 3rs + 1$, $W_{7,3} = 4r^2s + rs$, and $W_{7,4} = 4r^2s^2 + rs$, each contribute exactly one unique weight value because the labeling is constructed so that the configurations of incident edge labels on the corresponding vertex types do not overlap. Consequently, the cardinalities of these weight sets satisfy $|W_{7,1}| = 1$,

$|W_{7,2}| = 1$, $|W_{7,3}| = 1$, and $|W_{7,4}| = 1$, and the global cardinality of the vertex-weight set is obtained by summing these values, namely $|W| = |W_{7,1}| + |W_{7,2}| + |W_{7,3}| + |W_{7,4}| = 1 + 1 + 1 + 1 = 4$. This result shows that the labeling produces four distinct vertex weights, thereby satisfying the definition of a rainbow vertex antimagic labeling. Since the construction yields both the lower bound $rvac \geq 4$ and the upper bound $rvac \leq 4$, it follows that $rvac(\text{Amal}(D_r, z, s)) = 4$ for all $s \geq 2$ and $r \geq 2$. □

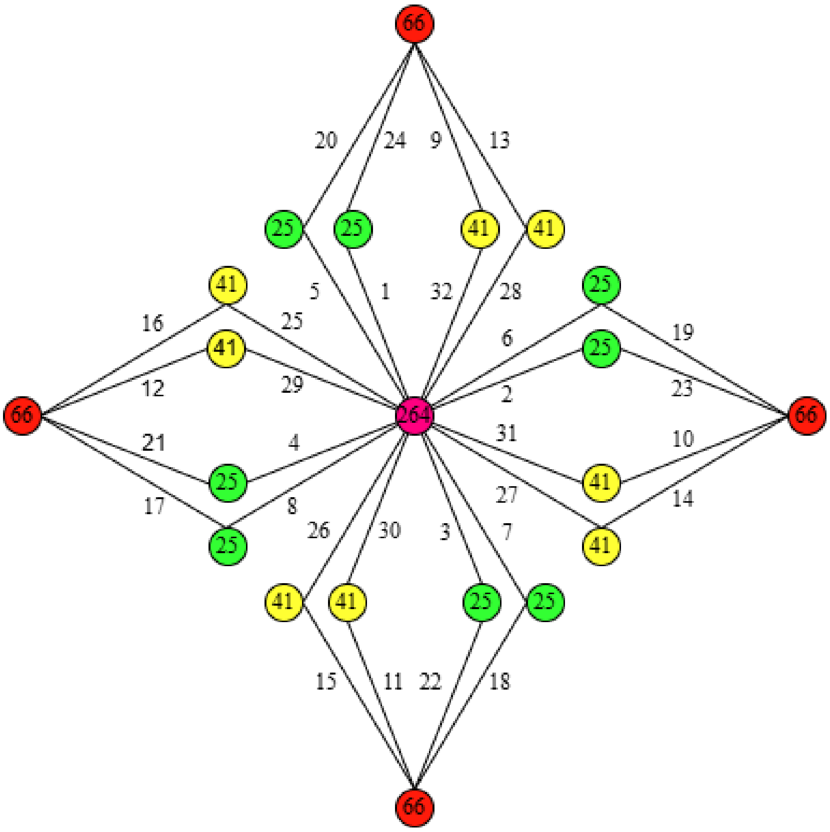

Figure 6. RVAC of $Amal(D_2, z, 4)$ graph.

The rainbow path of the graph $Amal(D_2, z, s)$ can be seen in Table 3 and an illustration of the rainbow vertex coloring of $Amal(D_2, z, s)$ can be seen in Figure 6.

Table 3. The Rainbow Path u–v Graph $Amal(D_r, z, s)$

u	v	**Rainbow path** u-v	**Condition**
z_t	z_{t+1}	$z_t, x_{t,q},\ z,\ y_{t+1,q},\ z_{t+1}$	$1 \leq t \leq s, 1 \leq q \leq r$
z_t	$y_{t+1,q}$	$z_t, x_{t,q},\ z,\ y_{t+1,q}$	$1 \leq t \leq s, 1 \leq q \leq r$
$y_{t,q}$	$y_{t,q+1}$	$y_{t,q},\ z,\ y_{t,q+1}$	$1 \leq t \leq s, 1 \leq q \leq r$

4. **Practical Applications.** Although this study is theoretical in nature, the obtained results on rainbow vertex antimagic coloring may be interpreted in several practical settings. In particular, the exact values of $rvac(G)$ can provide a mathematical basis for designing structured and distinguishable connection patterns in systems represented by modular graphs.

One possible application arises in modular communication or sensor networks. In such systems, several repeated local units are often connected through a common central node, forming a hub-based topology similar to an amalgamation graph. Under the RVAC framework, an antimagic edge labeling induces distinct vertex-weights, and a rainbow vertex path ensures that the internal vertices along the path have pairwise distinct weights. This property may be interpreted as a vertex-based route signature, so that different communication paths can be distinguished through the sequence of internal vertex-weights. Hence, RVAC may support route identification, traceability, and fault localization in modular network structures.

Another potential application is in document authentication and digital watermarking. A rainbow vertex path in an RVAC graph produces a deterministic sequence of distinct internal vertex-weights, which may be encoded as a structured verification pattern. Such a pattern can be embedded into a digital image or electronic document as a watermark or authentication code. During verification, the extracted pattern can be compared with the original RVAC-based sequence to detect possible modification or tampering. From this perspective, the exact values of $rvac(G)$ describe the minimum diversity of vertex-based patterns available on a given graph structure, which is useful in designing verifiable and non-repetitive authentication schemes.

Therefore, even though the main contribution of this paper is theoretical, the results have potential relevance in the modeling of modular communication systems and in the development of graph-based digital authentication methods.

5. **Concluding Remarks.** We have developed three new theorems on rainbow vertex connection and rainbow vertex antimagic coloring for specific graph structures, namely amalgamation graphs, including the Amalgamation Cuttlefish graph $Amal(CF_r, z, s)$, the Amalgamation Semi Jellyfish graph $Amal(SJ_r, z, s)$, and Amalgamation Diamond graph $Amal(D_r, z, s)$. Using bijective edge label constructions to induce vertex weights, we established exact values for both the rainbow vertex connection number (rvc) and the rainbow vertex antimagic connection number ($rvac$). The $rvc(Amal(CF_r, z, s))$ is $s + 1$ for $s \geq 3, r \geq 3$, $rvac(Amal(CF_r, z, s))$ is $sr+1$ for $s \geq 3, r \geq 3$, $rvc(Amal(SJ_r, z, s)) = s+1$ for $s \geq 3, r \geq 3$, $rvac(Amal(SJ_r, z, s))$ is $sr + 1$ for $s \geq 3, r \geq 3$, $rvc(Amal(D_r, z, s)) = 3$ for $s \geq 2, r \geq 2$, $rvac(Amal(D_r, z, s)) = 4$ for $s \geq 2, r \geq 2$. These results are obtained via explicit edge-label designs, vertex weight computations, and verification of representative rainbow paths. The constructions provide a template that can be adapted to other amalgamation graph families in future work. This study has several limitations. First, the results are derived only for three specific amalgamation graph classes with regular repeated structures. Second, the proofs rely on explicit constructive labelings, so the method is not yet algorithmic for arbitrary graph classes. Third, the obtained formulas hold only under the stated parameter conditions on r and s. Therefore, further work is needed to extend RVAC analysis to broader families of modular graphs and to develop more general construction methods.

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