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On Rainbow Antimagic Coloring of Amalgamation of Wheel Graph Families

Syabana Nurin Nadia Aziz[1], Dafik[2,3,*], Arika Indah Kristiana[3,4], Dian Kurniati[4], and N. Mohanapriya[5]

[1]Department of Postgraduate of Mathematics Education, University of Jember, Indonesia
[2]Department of Mathematics, University of Jember, Indonesia
[3]PUI-PT Combinatorics and Graph, CGANT-University of Jember, Indonesia
[4]Department of Mathematics Education, University of Jember, Indonesia
[5]PG and Research Department of Mathematics, Kongunadu Arts and Science College, India

*Corresponding author: d.dafik@unej.ac.id

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Abstract. *Graph coloring is a central topic in graph theory with numerous applications in scheduling, frequency assignment, and optimization. In this paper, we study rainbow antimagic coloring, which combines antimagic labeling and rainbow connection, and focus on the rainbow antimagic connection number* $rac(G)$ *for amalgamations of sunflower, lemon, and double wheel graphs. Using bijective vertex labelings that induce distinct edge-weights, we establish exact formulas for these graph families. In particular, we show that* $rac(G) = 3mn$ *for the sunflower amalgamation, while* $rac(G) = 2mn$ *for the lemon and double wheel amalgamations. Moreover, the induced edge-weights are pairwise distinct along suitable connections, guaranteeing the existence of rainbow paths between any two vertices. These results advance the theory of graph labeling and coloring and provide a basis for further studies on rainbow antimagic structures in more complex graphs. We also briefly discuss a practical implication of these exact values in designing RAC-derived watermarks for Police Clearance Certificate images using an LSB-based embedding and extraction verification scheme.*
Keywords: Rainbow Antimagic Coloring, Graph Theory, Rainbow Antimagic Connection Number.

1 Introduction

Graph coloring has long been a central topic in graph theory due to its broad range of applications in scheduling, frequency assignment, and optimization. The classical concept began with vertex coloring, in which adjacent vertices are assigned different colors to ensure clarity in graph structures. This foundational theory was significantly advanced by Kenneth Appel and Wolfgang Haken [1] in their computer-assisted proof of the Four Color Theorem, which demonstrated that any planar graph can be colored with no more than four colors such that no two adjacent vertices share the same color [2]. Since then, graph coloring has evolved into a wide area of research that encompasses both theoretical developments and application-driven formulations.

Expanding beyond vertex coloring, Hartsfield and Ringel introduced the concept of antimagic labeling in 1990 [3]. In this framework, integers are assigned to edges so that each vertex obtains a unique incident edge-sum. Formally, an antimagic labeling of a graph $G = (V, E)$ is a bijection $f : E \longrightarrow \{1, 2, \ldots, |E|\}$ such that the vertex sums $w(v) = \Sigma f(e_i)$ (for all e_i incident to v) are pairwise distinct. This sparked extensive interest in labeling problems that balance uniqueness conditions with combinatorial structure and constructive feasibility [4].

Another influential development appeared in the study of edge-colored connectivity. In 2008, Chartrand et al. introduced the concept of rainbow connection [5]. A connected graph is rainbow connected if any two vertices are connected by at least one path whose edges all have distinct colors. This notion introduced a refined measure of connectivity based on color diversity and motivated studies on how to ensure global connectivity under constraints on edge coloring.

Motivated by these parallel lines of research, Dafik et al. proposed a concept referred to as rainbow antimagic coloring (RAC) [6], which integrates antimagic-style uniqueness with rainbow connectivity. In the RAC framework, vertex labels are assigned bijectively and used to induce edge-weights, and a path is considered rainbow if all induced edge-weights along the path are distinct. The minimum number of colors (or induced-weight diversity) required so that for any two vertices there exists a rainbow path is known as the rainbow antimagic connection number, denoted by $rac(G)$ [7, 8]. A general lower bound commonly used in RAC studies is given as follows.

Lemma 1. [6] *Let G be any connected graph. Let $rc(G)$ and $\Delta(G)$ represent the rainbow connection number of G and the maximum degree of G, respectively. Then, it holds that $rac(G) \geq \max\{rc(G), \Delta(G)\}$.*

In recent years, a growing body of work has established exact values of $rac(G)$ for many structured graph families, mainly through explicit bijective constructions that control induced edge-weights and guarantee rainbow paths. Septory et al. [6] determined rainbow antimagic coloring of several special graphs, including jahangir, lemon, firecracker, complete bipartite, and double star graphs, thereby providing constructive patterns for computing $rac(G)$ under diverse degree configurations. Sulistiyono et al. [9] determined rainbow antimagic coloring of ladder, triangular ladder, and diamond graphs, extending RAC analysis to layered and diamond-like structures with repeated local patterns. Budi et al. [10] determined rainbow antimagic coloring of lollipop, stacked book, Dutch windmill, flowerpot, and dragonfly graphs, which enriches the catalogue of RAC results for families combining cycles and pendant attachments. Al Jabbar et al. [11] determined rainbow antimagic coloring of related book graphs, showing that book-type compositions also admit systematic RAC constructions.

Joedo et al. [12] determined rainbow antimagic coloring of vertex amalgamation of path, star, broom, paw, fan, and triangular book graphs, emphasizing how a shared vertex influences global labeling consistency across modules. Mursyidah et al. [13] determined rainbow antimagic coloring of vertex amalgamation of fan, dragon, bow, and wheel graphs, further clarifying RAC behavior on hub-based amalgamations with rich cyclic substructures. Lestari et al. [14] investigated strong rainbow antimagic coloring of jahangir, semi-jahangir, friendship, fan, star, cycle, and path graphs, highlighting more restrictive variants and their implications for exact values. Annadhifi et al. [15] determined rainbow antimagic coloring of double quadrilateral windmill, flower, split star, and tunjung graphs, adding additional exact results for windmill- and star-related configurations. Meganingtyas et al. [16] determined rainbow antimagic coloring of amalgamations involving jahangir, flowerpot, bull, and volcano graphs, indicating that RAC can be systematically studied under amalgamation operations. Maghfiro et al. [17] determined rainbow antimagic coloring of double wheel and parachute graphs, contributing results for wheel-based families with dense local connectivity. Septory et al. [18] studied rainbow antimagic coloring of corona products in connection numbers, linking RAC to graph product operations that expand graph structure in a controlled manner. Adawiyah et al. [19] determined rainbow antimagic coloring of snail, coconut root, fan stalk, and lotus graphs, broadening RAC results for graphs with branching and cycle features. Septory et al. [20] studied rainbow antimagic coloring of the comb product of friendship and tree graphs, showing that comb operations also yield tractable RAC constructions. Finally, Jannah et al. [21] investigated rainbow antimagic coloring and its application on multi-step time series forecasting for electronic traffic law enforcement, demonstrating that RAC can be connected to application-oriented contexts beyond purely theoretical graph classes.

Despite this progress, results for amalgamation-based constructions remain comparatively limited, especially for families that combine hub-based attachment with cycle-rich local structures. The amalgamation operation introduces additional constraints that do not arise in single-module graphs: a shared linking vertex couples the labeling across multiple modules, and the distinctness of induced edge-weights must hold globally, not only within each module. Consequently, determining an exact value of $rac(G)$ for an amalgamated family requires a careful balance between (i) constructing bijective labelings that induce pairwise distinct edge-weights across the entire graph and (ii) guaranteeing the existence of rainbow paths that may traverse different modules through the linking vertex. These considerations motivate the present work.

In this paper, we investigate the rainbow antimagic connection number $rac(G)$ for amalgamations of sunflower, lemon, and double wheel graphs. These families naturally represent modular topologies formed by attaching repeated subgraphs through a common linking vertex and provide a unified setting in which the interaction between induced edge-weight distinctness and rainbow connectivity can be analyzed systematically. Our main contribution is to derive closed-form expressions for $rac(G)$ for these amalgamated families via explicit bijective labeling constructions, to prove the distinctness of the induced edge-weights, and to exhibit rainbow paths connecting any two vertices. In addition, we briefly discuss a practical implication of the obtained formulas by illustrating how RAC-derived induced edge-weight patterns may be encoded as verifiable watermarks for Police Clearance Certificate images under a Least Significant Bit (LSB) embedding and extraction framework.

2 Methods This research employs a deductive analytical approach to determine the rainbow antimagic coloring of graphs. The process begins by defining a graph $\mathbb{G} = (V, E)$, where V represents the set of vertices and E represents the set of edges. In rainbow antimagic coloring, each edge of the graph is assigned a unique weight, and each vertex is labeled with a distinct integer. Specifically, a bijective labeling function $f : V \to \{1, 2, 3, \ldots, |V|\}$ is applied to the vertices, ensuring that each vertex is assigned a unique integer label. Then, the weight of each edge $uv \in E$ is computed as the sum of the labels of its incident vertices, i.e., $w(uv) = f(u) + f(v)$, ensuring that no two edges share the same weight.

The primary objective of rainbow antimagic coloring is to guarantee that the graph is rainbow-connected, meaning that for every pair of vertices u and v, there exists at least one path between them such that all edges on the path have distinct weights. Moreover, the sum of the weights along any two distinct paths between u and v must be different, ensuring that the paths do not share the same rainbow weight sum. The rainbow antimagic connection number, denoted by $rac(G)$, is defined as the minimum number of colors required to ensure rainbow connectivity for all paths in the graph.

To determine the rainbow antimagic coloring, the method proceeds by constructing a weight assignment for the edges and ensuring that the edge weights are distinct. The graph is then evaluated for its rainbow connectivity, where paths between every pair of vertices are examined to ensure that the edge weights are distinct and that the sum of the edge weights along any two distinct paths are different. The final result is the rainbow antimagic connection number, which provides the minimum number of colors required for the graph to be rainbow-connected. This method is applied to various families of graphs to analyze the properties of rainbow antimagic coloring and derive new theoretical bounds for the rainbow antimagic connection number.

3 Main Results In this paper, we discuss some new results of the rainbow antimagic coloring of amalgamation of graphs.

Theorem 1. *Let $Amal(Sf_n, v, m)$ be an amalgamation of sunflower graphs, with the central vertex as the linking vertex. For every positive integer $n \geq 3$, we have $rac(Amal(Sf_n, v, m)) = 3mn$.*

Proof. $Amal(Sf_n, v, m)$ is amalgamation of sunflower graphs, with the central vertex as the linking vertex with vertex set $V(Amal(Sf_n, v, m)) = \{x_{i,j}, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{y_{i,j}, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{z_{i,j}, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{v\}$ and edge set $E(Amal(Sf_n, v, m)) = \{x_{i,j}v, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{y_{i,j}v, 1 \leq i \leq n \& 1 \leq i \leq m\} \cup \{z_{i,j}v, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{x_{i,j}y_{i,j}, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{x_{i,j}x_{i+1,j}, 1 \leq i \leq n-1 \ \& \ 1 \leq i \leq m\} \cup \{x_{i=n,j}x_{1,j}, 1 \leq i \leq m\}$. The order and size of $Amal(Sf_n, v, m)$ are $3mn + 1$ and $5mn$, respectively. The maximum degree of $Amal(Sf_n, v, m)$ is $3mn$.

To prove that $rac(Amal(Sf_n, v, m)) = 3mn$, we must establish both the lower bound, $rac(Amal(Sf_n, v, m)) \geq 3mn$, and the upper bound, $rac(Amal(Sf_n, v, m)) \leq 3mn$. Based on Lemma 1, we have $rac(Amal(Sf_n, v, m)) \geq \max\{rc(Amal(Sf_n, v, m)), \Delta(Amal(Sf_n, v, m))\} = \max\{mn, 3mn\} = 3mn$. Next, we show the upper bound $rac(Amal(Sf_n, v, m)) \leq 3mn$ using the following bijective functions.

$$
\begin{aligned}
f(x_{i,j}) &= nj - n + i + 1, && \text{for } 1 \leq i \leq n,\ 1 \leq j \leq m \\
f(y_{i,j}) &= nm + jn - n + i + 1, && \text{for } 1 \leq i \leq n,\ 1 \leq j \leq m \\
f(z_{i,j}) &= 2nm + jn - n + i + 1, && \text{for } 1 \leq i \leq n,\ 1 \leq j \leq m \\
f(v) &= 1
\end{aligned}
$$

Based on a vertex function, the edge weight is as follows:

$$\begin{aligned}
w(x_{i,j}v) &= nj - n + i + 2, && \text{for } 1 \le i \le n,\ 1 \le j \le m\\
w(y_{i,j}v) &= nm + nj - n + i + 2, && \text{for } 1 \le i \le n,\ 1 \le j \le m\\
w(z_{i,j}v) &= 2nm + jn - n + i + 2, && \text{for } 1 \le i \le n,\ 1 \le j \le m\\
w(x_{i,j}y_{i,j}) &= nm + 2jn - 2n + 2i + 2, && \text{for } 1 \le i \le n,\ 1 \le j \le m\\
w(x_{i,j}x_{i+1,j}) &= 2i + 2jn - 2n + 3, && \text{for } 1 \le i \le n-1,\ 1 \le j \le m\\
w(x_{n,j}x_{1,j}) &= 2jn - n + 3, && \text{for } 1 \le j \le m
\end{aligned}$$

The edge weight above will induce the rainbow antimagic coloring of graph, such that the number of different edge weight as follows:

$W_1 = \{w(x_{i,j}v)\} = \{3, 4, 5, 6, \ldots, mn + 2\}$

$$U_{|W_1|} = a + (|W_1| - 1)b$$

$$\begin{aligned}
mn + 2 &= 3 + (|W_1| - 1) \cdot 1\\
mn + 2 &= 2 + |W_1|\\
|W_1| &= mn + 2 - 2 = mn
\end{aligned}$$

$W_2 = \{\, w(y_{i,j}, v) \,\} = \{\, mn + 3,\ mn + 4,\ mn + 5,\ \ldots,\ 2mn + 2 \,\}$

$$U_{|W_2|} = a + (|W_2| - 1)b$$

$$\begin{aligned}
2mn + 2 &= (mn + 3) + (|W_2| - 1) \cdot 1\\
2mn + 2 &= mn + |W_2| + 2\\
|W_2| &= 2mn + 2 - mn - 2 = mn
\end{aligned}$$

$W_3 = \{\, w(z_{i,j}, v) \,\} = \{\, 2mn + 3,\ 2mn + 4,\ 2mn + 5,\ \ldots,\ 3mn + 2 \,\}$

$$U_{|W_3|} = a + (|W_3| - 1)b$$

$$\begin{aligned}
3mn + 2 &= (2mn + 3) + (|W_3| - 1) \cdot 1\\
3mn + 2 &= 2mn + |W_3| + 2\\
|W_3| &= 3mn + 2 - 2mn - 2 = mn
\end{aligned}$$

We now show that the intersection of the sets W_1, W_2, and W_3 is $W_1 \cap W_2 \cap W_3 = \emptyset$. To prove that the intersection of these three sets is empty, we first need to analyze the structure of each set and show that there are no common elements between them.

For W_1, the weight is given by $w_1(i, j) = nj - n + i + 2$. The minimum value of W_1 occurs when $i = 1$ and $j = 1$, giving $n(1) - n + 1 + 2 = 3$. The maximum value of W_1 occurs when $i = n$ and $j = m$, giving $nm - n + n + 2 = nm + 2$. Thus, the range of W_1 is $[3, nm + 2]$. For W_2, the weight is given by $w_2(i, j) = nm + nj - n + i + 2$. The minimum value occurs when $i = 1$ and $j = 1$, giving $nm + n(1) - n + 1 + 2 = nm + 3$. The maximum value occurs when $i = n$ and $j = m$, giving $nm + nm - n + n + 2 = 2nm + 2$. Thus, the range of W_2 is $[nm + 3, 2nm + 2]$. For W_3, the weight is given by $w_3(i, j) = 2nm + jn - n + i + 2$. The minimum value occurs when $i = 1$ and $j = 1$, giving $2nm + n(1) - n + 1 + 2 = 2nm + 3$. The maximum value occurs when $i = n$ and $j = m$, giving $2nm + nm - n + n + 2 = 3nm + 2$. Thus, the range of W_3 is $[2nm + 3, 3nm + 2]$.

Now, to show that the intersection $W_1 \cap W_2 \cap W_3 = \emptyset$, we examine the ranges of W_1, W_2, and W_3. The range of W_1 is $[3, nm+2]$, the range of W_2 is $[nm+3, 2nm+2]$, and the range of W_3 is $[2nm+3, 3nm+2]$. It is evident that there is no overlap between these ranges: the maximum value of W_1 is $nm+2$, while the minimum value of W_2 is $nm + 3$, which is strictly greater than the maximum value of W_1. Similarly, the maximum value of W_2 is $2nm + 2$, while the minimum value of W_3 is $2nm + 3$, which is strictly greater than the maximum value of W_2. Moreover, the range of W_3 starts at $2nm + 3$, which is strictly greater than the maximum value of W_2, and extends up to $3nm + 2$, which is strictly greater than the maximum value of W_1.

Since there is no overlap between the ranges of W_1, W_2, and W_3, the intersection $W_1 \cap W_2 \cap W_3$ is indeed empty. Therefore, we conclude that:

$$W_1 \cap W_2 \cap W_3 = \emptyset.$$

We are tasked with proving that that $W_4 \subseteq W_2 \cup W_3$ under the assumption that $1 \leq i \leq n$ and $1 \leq j \leq m$. Now consider W_4, where $w_4(i,j) = nm+2jn-2n+2i+2 = nm+2n(j-1)+2i+2$. The minimum value of $w_4(i,j)$ is obtained when $i=1$ and $j=1$, yielding $W_4^{\min} = nm+2n(1-1)+2(1)+2 = nm+4$,while the maximum value appears when $i=n$ and $j=m$, namely $W_4^{\max} = nm+2n(m-1)+2n+2 = 3nm+2$. Thus every element of W_4 lies in the integer interval $W_4 \subseteq [\,nm+4,\ 3nm+2\,] \cap \mathbb{Z}$. Since $[\,nm+4,\ 3nm+2\,] \cap \mathbb{Z} \subseteq [\,nm+3,\ 3nm+2\,] \cap \mathbb{Z} = W_2 \cup W_3$, it follows that each weight in W_4 is already realized as a vertex-weight in either W_2 or W_3. Hence we conclude that

$$W_4 \subseteq W_2 \cup W_3$$

. We are tasked with proving that $W_5 \subseteq W_1 \cup W_2$. We analyze W_5, where the weight is given by $w_5(i,j) = 2i+2jn-2n+3$. The minimum value occurs when $i=1$ and $j=1$, resulting in $W_5^{\min} = 2(1)+2n(1)-2n+3 = 5$. The maximum value occurs when $i=n$ and $j=m$, which gives $W_5^{\max} = 2n+2nm-2n+3 = 2nm+3$. Thus, the range of W_5 is the interval $[5, 2nm+3]$, containing all integer values from 5 up to $2nm+3$.

Now, we verify that $W_5 \subseteq W_1 \cup W_2$. The range of W_5 is $[5, 2nm+3]$, which begins at 5 and ends at $2nm+3$. The range of W_1 is $[3, nm+2]$, and the range of W_2 is $[nm+3, 2nm+2]$. Clearly, the values in the range of W_5, from 5 to $2nm+3$, are covered by the union of the ranges of W_1 and W_2, as the lower bound of W_5 (which is 5) is greater than or equal to the lower bound of W_1 (which is 3), and the upper bound of W_5 (which is $2nm+3$) is greater than or equal to the upper bound of W_2 (which is $2nm+2$).

Therefore, every element of W_5 is contained in either W_1 or W_2, and we conclude that

$$W_5 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_6 \subseteq W_1 \cup W_2$, for W_6, the weight is $w_6(j) = 2jn-n+3$. The minimum value occurs when $j=1$, yielding $W_6^{\min} = n+3$, and the maximum value occurs when $j=m$, resulting in $W_6^{\max} = 2nm-n+3$. Therefore, the range of W_6 is $[n+3, 2nm-n+3]$.

Now, we show that $W_6 \subseteq W_1 \cup W_2$. The range of W_6 is $[n+3, 2nm-n+3]$, which starts at $n+3$ and ends at $2nm-n+3$. The range of W_1 is $[3, nm+2]$, and the range of W_2 is $[nm+3, 2nm+2]$. Notice that the lower bound of W_6 is $n+3$, which is greater than or equal to the lower bound of W_1 (which is 3) and falls within the range of W_2. The upper bound of W_6 is $2nm-n+3$, which is less than or equal to the upper bound of W_2 (which is $2nm+2$). Therefore, every element of W_6 is contained within either W_1 or W_2, meaning that

$$W_6 \subseteq W_1 \cup W_2.$$

In each of the above sets, it can be observed that the smallest and largest values fall within the interval $[3, 3mn+2]$. Therefore, the number of distinct edge weights is $|W_1 \cup W_2 \cup W_3| = |W_1|+|W_2|+|W_3| = mn + mn + mn = 3mn$. This implies that $rac(Amal(Sf_n, v, m)) \leq 3mn$. Since $rac(Amal(Sf_n, v, m)) \geq 3mn$ and $rac(Amal(Sf_n, v, m)) \leq 3mn$, we conclude that $rac(Amal(Sf_n, v, m)) = 3mn$.

For any two vertices, there exists a rainbow path, as presented in Table 1.

TABLE 1. $y - z$ Rainbow path in amalgamation of sunflower graph $Amal(Sf_n, v, m)$.

y	**z**	**rainbow path**
$x_{i,j}$	v	$x_{i,j}, v$
$y_{i,j}$	v	$y_{i,j}, v$
$z_{i,j}$	v	$z_{i,j}, v$
$x_{i,j}$	$y_{i,j}$	$x_{i,j}, v, y_{i,j}$
$x_{i,j}$	$z_{i,j}$	$x_{i,j}, v, z_{i,j}$
$y_{i,j}$	$z_{i,j}$	$y_{i,j}, v, z_{i,j}$
$x_{i,j}$	$x_{k,l}$	$x_{i,j}, v, x_{k,l}$
$y_{i,j}$	$y_{k,}$	$y_{i,j}, v, y_{k,l}$
$z_{i,j}$	$z_{k,l}$	$z_{i,j}, v, z_{k,l}$
$x_{i,j}$	$y_{k,l}$	$x_{i,j}, v, y_{k,l}$
$x_{i,j}$	$z_{k,l}$	$x_{i,j}, v, y_{k,l}$
$y_{i,j}$	$z_{k,l}$	$x_{i,j}, v, y_{k,l}$

It can be easily observed from the table above that for any two vertices, there exists a rainbow path between them, which satisfies the condition for a valid rainbow antimagic coloring. □

As an example can be viewed in Figure 1.

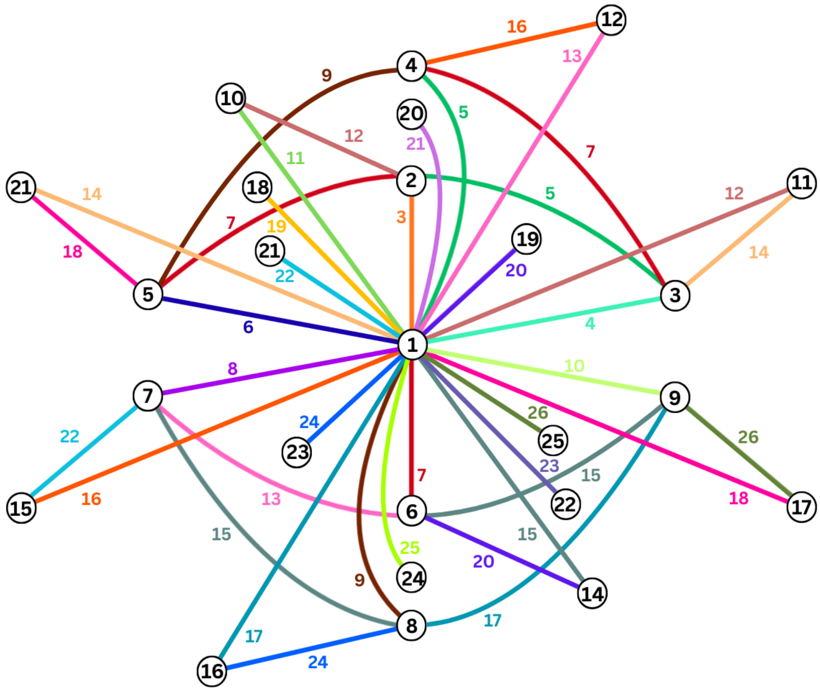

FIGURE 1. A rainbow antimagic coloring of $rac(Amal(Sf_4, v, 2))$

Theorem 2. *Let $Amal(Le_n, v, m)$ be a amalgamation of lemon graph, with the central vertex as the linking vertex. For every positive integer $n \geq 3$. We have $rac(Amal(Le_n, v, m)) = 2mn$.*

Proof. $Amal(Le_n, v, m)$ is amalgamation of lemon graph with the central vertex as the linking vertex with vertex set $V(Amal(Le_n, v, m)) = \{x_{i,j}, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{y_{i,j}, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{v\}$ and edge set $E(Amal(Le_n; v, m)) = \{x_{i,j}v, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{y_{i,j}v, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{x_{i,j}y_{i,j}, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{x_{i,j}x_{i+1,j}, 1 \leq i \leq n-1 \ \& \ 1 \leq i \leq m\} \cup \{x_{i=n,j}x_{1,j}, 1 \leq i \leq m\} \cup \{y_{i,j}y_{i+1,j}, 1 \leq i \leq n-1 \ \& \ 1 \leq i \leq m\} \cup \{y_{i=n,j}y_{1,j}, 1 \leq i \leq m\}$. The order and size of $Amal(Le_n, v, m)$ are $2mn+1$ and $5mn$, respectively. The maximum degree of $Amal(Le_n, v, m)$ is $2mn$.

To prove that $rac(Amal(Le_n, v, m)) = 2mn$, we must establish both the lower bound, $rac(Amal(Le_n, v, m)) \geq 2mn$, and the upper bound, $rac(Amal(Le_n, v, m)) \leq 2mn$. Based on Lemma 1, we have $rac(Amal(Le_n, v, m)) \geq max\{rc(Amal(Le_n, v, m))\Delta(Amal(Le_n, v, \ m))\} = max\{2mn, 2mn\} = 2mn$. Next, we show the upper bound $rac(Amal(Le_n, v, m)) \leq 2mn$ using the following bijective functions.

Case 1. for n =odd

$f(v) = m(n-1)+1$

$$f(x_{i,j}) = \begin{cases} m(n+1)+(m+j-1)\frac{n-1}{2}+\frac{i+1}{2}+1, & \text{if } i \text{ odd, for } 1 \leq i \leq n-2,\ 1 \leq j \leq m \\ i+(j-1)(n-1), & \text{if } i \text{ even, for } 2 \leq i \leq n-1,\ 1 \leq j \leq m \\ im-j+2, & \text{for } i = n,\ 1 \leq j \leq m \end{cases}$$

$$f(y_{i,j}) = \begin{cases} i+(n-1)(j-1+m)+1, & \text{if } i \text{ odd, for } 1 \leq i \leq n-2,\ 1 \leq j \leq m \\ mn+m+\dfrac{i}{2}+1+(j-1)\dfrac{n-1}{2}, & \text{if } i \text{ even, for } 2 \leq i \leq n-1,\ 1 \leq j \leq m \\ im+m-j+2, & \text{for } i = n,\ 1 \leq j \leq m \end{cases}$$

Based on a vertex function, the edge weight is as follows:

$$w(x_{i,j}v)=\begin{cases}2mn+(m+j-1)\frac{n-1}{2}+\frac{i+1}{2}+2, & \text{if } i \text{ odd},\ 1\le i\le n-2,\ 1\le j\le m\\ i+(j-1)n+m(n-1)-j+2, & \text{if } i \text{ even},\ 2\le i\le n-1,\ 1\le j\le m\\ im-j+mn-m+3, & \text{for } i=n,\ 1\le j\le m\end{cases}$$

$$w(y_{i,j}v)=\begin{cases}i+(j-1)n+m(n-1)-j+2, & \text{if } i \text{ odd},\ 1\le i\le n-2,\ 1\le j\le m\\ 2mn+\frac{i}{2}+1+(j-1)\frac{n-1}{2}, & \text{if } i \text{ even},\ 2\le i\le n-1,\ 1\le j\le m\\ im+mn-j+3, & \text{for } i=n,\ 1\le j\le m\end{cases}$$

$$w(x_{i,j}y_{i,j})=\begin{cases}m(\frac{5n-3}{2})+\frac{3i+5}{2}+j(\frac{5(n-1)}{2}), & \text{if } i \text{ odd},\ 1\le i\le n-2,\ 1\le j\le m\\ \frac{3i}{2}+\frac{3(j-1)(n-1)}{2}+m(n+1)+1, & \text{if } i \text{ even},\ 2\le i\le n-1,\ 1\le j\le m\\ 2im+2mn-2j-m+6, & \text{for } i=n,\ 1\le j\le m\end{cases}$$

$$w(x_{i,j}x_{i+1,j})=\begin{cases}n\left(\frac{3m}{2}+2j\right)-\frac{m}{2}+\frac{3i+7}{2}, & \text{if } i \text{ odd},\ 1\le i\le n-2,\ 1\le j\le m\\ mn+m+\frac{3i+4}{2}+\frac{(n-1)(3j+m-3)}{2}, & \text{if } i \text{ even},\ 1\le i\le n-1,\ 1\le j\le m\end{cases}$$

$$w(x_{n,j}x_{1,j})=m(\tfrac{5n-3}{2})+j(\tfrac{3n-5}{2})+4,\qquad 1\le j\le m$$

$$w(y_{i,j}y_{i+1,j})=\begin{cases}\frac{3i+1}{2}+\frac{(n-1)(3j-1)}{2}+m(n+1)+2, & \text{if } i \text{ odd},\ 1\le i\le n-2,\ 1\le j\le m\\ m(\frac{5n-3}{2})+j(\frac{3n-5}{2})+4, & \text{if } i \text{ even},\ 1\le i\le n-1,\ 1\le j\le m\end{cases}$$

$$w(y_{n,j}y_{1,j})=mn+j(n-2)+5-n,\qquad 1\le j\le m$$

The edge weight above will induce the rainbow antimagic coloring of graph, such that the number of different edge weight as follows:
$W_1=w(x_{i,j}v)=W_{1a}\cup W_{1b}\cup W_{1c}$, a for i odd $(1\le i<n-2)$, b for i even $(2\le i\le n-1)$, c for $i=n$

$$W_{1a}=\{\,2mn+\tfrac{m(n-1)}{2}+3,\ 2mn+\tfrac{m(n-1)}{2}+4,\dots,\ 2mn+\tfrac{n}{2}+\tfrac{(2m-1)(n-1)}{2}+\tfrac{3}{2}\}$$

$$U_{|W_{1a}|}=a+(|W_{1a}|-1)b$$
$$2mn+\tfrac{n}{2}+\tfrac{(2m-1)(n-1)}{2}+\tfrac{3}{2}=2mn+\tfrac{m(n-1)}{2}+3+(|W_{1a}|-1)\cdot 1$$
$$|W_{1a}|=\tfrac{(n-1)m}{2}$$

$$W_{1b}=\{\,m(n-1)+3,\,m(n-1)+5,\,\dots,\,(n-1)+(m-1)n+m(n-1)-m+2\,\}$$

$$U_{|W_{1b}|}=a+(|W_{1b}|-1)b$$
$$(n-1)+(m-1)n+m(n-1)-m+2=m(n-1)+3+(|W_{1b}|-1)\cdot 2$$
$$|W_{1b}|=\tfrac{(n-1)m}{2}$$

$$W_{1c}=\{\,2mn-m+2,\,2mn-m+1,\,\dots,\,2mn-2m+3\,\}$$

$$U_{|W_{1c}|}=a+(|W_{1c}|-1)b$$
$$2mn-2m+3=2mn-m+2+(|W_{1c}|-1)\cdot -1$$
$$|W_{1c}|=m$$
$$W_1=W_{1a}\cup W_{1b}\cup W_{1c},|W_1|=|W_{1a}|+|W_{1b}|+|W_{1c}|=\tfrac{(n-1)m}{2}+\tfrac{(n-1)m}{2}+m=mn$$

$W_2=w(y_{i,j}v)=W_{2a}\cup W_{2c}\cup W_{2b}$, a for i odd $(1\le i<n)$, b for i even $(2\le i\le n)$, c for $i=n$,

$$W_{2a}=\{\,m(n-1)+2,\ m(n-1)+4,\dots,\ (n-2)+(m-1)n+m(n-1)-m+2\,\}$$

$$U_{|W_{2a}|} = a + (|W_{2a}| - 1)b$$
$$(n-2) + (m-1)n + m(n-1) - m + 2 = m(n-1) + 2 + (|W_{2a}| - 1) \cdot 2$$
$$|W_{2a}| = \tfrac{(n-1)m}{2}$$
$$W_{2b} = \{\, 2mn+2,\ 2mn+3,\ 2mn+4,\ \ldots,\ 2mn + \tfrac{(n-1)m}{2} + 1 \,\}$$

$$U_{|W_{2b}|} = a + (|W_{2b}| - 1)b$$
$$2mn + \frac{(n-1)m}{2} + 1 = 2mn + 2 + (|W_{2b}| - 1) \cdot 1$$
$$|W_{2b}| = \tfrac{(n-1)m}{2}$$
$$W_{2c} = \{\, 2mn+2,\ 2mn+1,\ 2mn, \ldots,\ 2mn - m + 3 \,\}$$

$$U_{|W_{2c}|} = a + (|W_{2c}| - 1)b$$
$$2mn - m + 3 = 2mn + 2 + (|W_{2c}| - 1) \cdot -1$$
$$|W_{2c}| = m$$
$$W_2 = W_{2a} \cup W_{2b} \cup W_{2c}, |W_2| = |W_{2a}| + |W_{2b}| + |W_{2c}| = \tfrac{(n-1)m}{2} + \tfrac{(n-1)m}{2} + m = mn.$$

We aim to prove that the intersection of the sets W_1 and W_2 is empty. To demonstrate this, we will examine the structure of the weight functions defined for each set and show that there are no common elements between them.

First, we consider the case when i is odd, it is evident that the expressions for $W_1(i,j)$ and $W_2(i,j)$ are fundamentally different. The presence of the term $2mn$ in W_1 and the absence of this term in W_2 indicates that there cannot be any overlap between these sets for odd values of i. Next, we consider the case when i is even, with $2 \le i \le n-1$ and $1 \le j \le m$. For W_1, the expressions are distinct. The term $2mn$ in W_2 does not appear in W_1, and the additional fractional terms in W_2 further differentiate the two weight functions. Therefore, no element can belong to both sets when i is even. Next, we consider the case when $i = n$ and $1 \le j \le m$. The difference between these two expressions is the term $-m$ in W_1, which is absent in W_2. This discrepancy ensures that no element can belong to both sets when $i = n$. Since we have shown that there are no values of (i,j) that satisfy both $W_1(i,j)$ and $W_2(i,j)$ simultaneously, it follows that:

$$W_1 \cap W_2 = \emptyset.$$

Thus, we conclude that the intersection of W_1 and W_2 is indeed the empty set.

We are tasked with proving that $W_3 \subseteq W_1 \cup W_2$. To do so, we first calculate the minimum and maximum values of W_3 for each case. For the first case, when i is odd, $1 \le i \le n-2$, and $1 \le j \le m$, the minimum weight occurs when $i = 1$ and $j = 1$, giving $m\left(\frac{5n-3}{2}\right) + \frac{3(1)+5}{2} + 1\left(\frac{5(n-1)}{2}\right)$, while the maximum weight occurs when $i = n-2$ and $j = m$, giving $m\left(\frac{5n-3}{2}\right) + \frac{3(n-2)+5}{2} + m\left(\frac{5(n-1)}{2}\right)$. For the second case, when i is even, $2 \le i \le n-1$, and $1 \le j \le m$, the minimum value occurs when $i = 2$ and $j = 1$, which is $3 + m(n+1) + 1$, and the maximum value occurs when $i = n-1$ and $j = m$, giving $\frac{3(n-1)}{2} + \frac{3(m-1)(n-1)}{2} + m(n+1) + 1$. In the third case, for $i = n$ and $1 \le j \le m$, the minimum value occurs when $j = 1$, giving $2n \cdot m + 2mn - 2(1) - m + 6 = 2mn + 2mn - 2 - m + 6$, and the maximum value occurs when $j = m$, giving $2nm + 2mn - 2m - m + 6$. Thus, the range of values for W_3 is the interval $[W_3^{\min}, W_3^{\max}]$, where $W_3^{\min}$ and $W_3^{\max}$ are the minimum and maximum values calculated above, respectively. Now, we compare the range of W_3 with the ranges of W_1 and W_2. The range of W_1 is given by $[3, 3nm+2]$, and the range of W_2 is given by $[3, 2nm+2]$. Since the range of W_3, which lies within $[W_3^{\min}, W_3^{\max}]$, is fully contained within the union of W_1 and W_2, it follows that

$$W_3 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_4 \subseteq W_1 \cup W_2$. To begin, we calculate the minimum and maximum values for W_4 in both cases. For the first case, when i is odd and $1 \le i \le n-2, 1 \le j \le m$, the minimum weight occurs when $i = 1$ and $j = 1$, yielding $n\left(\frac{3m}{2} + 2\right) - \frac{m}{2} + 5$, and the maximum weight occurs when $i = n-2$ and $j = m$, yielding $n\left(\frac{3m}{2} + 2m\right) - \frac{m}{2} + \frac{3(n-2)+7}{2}$. For the second case, when i is even and $1 \le i \le n-1, 1 \le j \le m$, the minimum value occurs when $i = 2$ and $j = 1$, yielding $mn + m + \frac{10}{2} + \frac{(n-1)(m)}{2}$, and the maximum value occurs when $i = n-1$ and $j = m$, yielding $mn + m + \frac{3(n-1)+4}{2} + \frac{(n-1)(3m+m-3)}{2}$. Thus, the range of W_4 lies between the minimum value $W_4^{\min}$ and the maximum value $W_4^{\max}$, where these values are computed as the smallest and largest weights in each case. We then compare the range of W_4 with the ranges of W_1 and W_2. The range of W_1 is given by $[3, 3nm+2]$, and the range of W_2 is given by $[3, 2nm+2]$. By comparing these ranges, we observe that the range of W_4 lies within the combined range of $W_1 \cup W_2$, as the values of W_4 fall within the interval covered by W_1 and W_2. Therefore, we conclude that

$$W_4 \subseteq W_1 \cup W_2$$

We are tasked with proving that $W_5 \subseteq W_1 \cup W_2$. To begin, we examine the weight function for W_5. The minimum value occurs when $j = 1$, giving $w_5(1) = m\left(\frac{5n-3}{2}\right) + \frac{3n-5}{2} + 4 = \frac{5mn-3m+3n-5}{2} + 4 = \frac{5mn-3m+3n+3}{2}$. Thus, the minimum value of w_5 is $\frac{5mn-3m+3n+3}{2}$. The maximum value occurs when $j = m$, which gives $w_5(m) = m\left(\frac{5n-3}{2}\right) + m\left(\frac{3n-5}{2}\right) + 4 = m\left(\frac{8n-8}{2}\right) + 4 = 4mn - 4m + 4$. Thus, the maximum value of w_5 is $4mn - 4m + 4$.

Now, we compare the range of W_5, which is the interval $[W_5^{\min}, W_5^{\max}]$, with the ranges of W_1 and W_2. The range of W_1 is $[3, 3nm+2]$, and the range of W_2 is $[3, 2nm+2]$. By comparing these ranges, we see that the minimum value of W_5, which is $W_5^{\min} = \frac{5mn-3m+3n+3}{2}$, is greater than or equal to 3, and the maximum value $W_5^{\max} = 4mn - 4m + 4$ lies within the upper bounds of both W_1 and W_2. Therefore, the entire range of W_5 is contained within the range of $W_1 \cup W_2$. Hence, we conclude that

$$W_5 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_6 \subseteq W_1 \cup W_2$. To begin, we examine the weight function for W_6. First, we analyze the minimum and maximum values for each case. For the case when i is odd, $1 \le i \le n-2$, and $1 \le j \le m$, the minimum value occurs when $i = 1$ and $j = 1$, which gives $w_6(1,1) = \frac{3(1)+1}{2} + \frac{(n-1)(3(1)-1)}{2} + m(n+1) + 2 = 2 + (n-1) + m(n+1) + 2 = n + m(n+1) + 4$. Thus, the minimum value of w_6 for this case is $n + m(n+1) + 4$. The maximum value occurs when $i = n-2$ and $j = m$, which gives an expression for the maximum weight, though its exact value is derived through algebraic simplifications.

For the case when i is even, $1 \le i \le n-1$, and $1 \le j \le m$, the minimum value occurs when $i = 2$ and $j = 1$, yielding $w_6(2,1) = m\left(\frac{5n-3}{2}\right) + \frac{3n-5}{2} + 4 = \frac{5mn-3m+3n-5}{2} + 4 = \frac{5mn-3m+3n+3}{2}$. Thus, the minimum value of w_6 for this case is $\frac{5mn-3m+3n+3}{2}$. The maximum value occurs when $i = n-1$ and $j = m$, yielding $w_6(n-1, m) = 4mn - 4m + 4$. Thus, the maximum value of w_6 for this case is $4mn - 4m + 4$.

Now, we compare the range of W_6 with the ranges of W_1 and W_2. The range of W_1 is $[3, 3nm+2]$, and the range of W_2 is $[3, 2nm+2]$. From the calculations above, we observe that the minimum value $W_6^{\min} = n + m(n+1) + 4$ is greater than or equal to 3, and the maximum value $W_6^{\max} = 4mn - 4m + 4$ lies within the upper bounds of both W_1 and W_2. Therefore, the entire range of W_6 is contained within the range of $W_1 \cup W_2$. Thus, we conclude that

$$W_6 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_7 \subseteq W_1 \cup W_2$. To begin, we examine the weight function for W_7. The minimum value occurs when $j = 1$, yielding $w_7(1) = mn + (1)(n-2) + 5 - n = mn + (n-2) + 5 - n = mn + 3 - 2 = mn + 1$. Thus, the minimum value of w_7 is $mn + 1$. The maximum value occurs when $j = m$, which gives $w_7(m) = mn + m(n-2) + 5 - n = mn + m(n-2) + 5 - n = 2mn - 2m + 5 - n$. Thus, the maximum value of w_7 is $2mn - 2m + 5 - n$.

Next, we compare the range of W_7, which is the interval $[W_7^{\min}, W_7^{\max}]$, with the ranges of W_1 and W_2. The range of W_1 is $[3, 3nm+2]$, and the range of W_2 is $[3, 2nm+2]$. Since the minimum value $W_7^{\min} = mn + 1$ is greater than or equal to 3, it falls within the lower bound of the range for both W_1 and W_2. Furthermore, the maximum value $W_7^{\max} = 2mn - 2m + 5 - n$ lies within the upper bound of $W_1 \cup W_2$, as it is less than or equal to $3nm + 2$ for suitable values of m and n. Therefore, the entire range of W_7 is contained within the union of W_1 and W_2. Thus, we conclude that

$$W_7 \subseteq W_1 \cup W_2.$$

Therefore, the number of distinct edge weights is $|W_1| + |W_2| = mn + mn = 2mn$. This implies that $rac(Amal(Le_n; v, m)) \le 2mn$. Since $rac(Amal(Le_n; v, m)) \ge 2mn$ and $rac(Amal(Le_n; v, m)) \le 2mn$, we conclude that $rac(Amal(Le_n; v, m)) = 2mn$.

Case 2. for n =even

$f(v) = mn + 1$

$$f(x_{i,j}) = \begin{cases} n(m+j-1)+i+1, & \text{if } i \text{ odd, for } 1 \le i \le n-1,\ 1 \le j \le m \\ i+(j-1)n, & \text{if } i \text{ even, for } 2 \le i \le n,\ 1 \le j \le m \end{cases}$$

$$f(y_{i,j}) = \begin{cases} i+(j-1)n, & \text{if } i \text{ odd, for } 1 \le i \le n-1,\ 1 \le j \le m \\ n(m+j-1)+i+1, & \text{if } i \text{ odd, for } 2 \le i \le n,\ 1 \le j \le m \end{cases}$$

Based on a vertex function, the edge weight is as follows:

$$w(x_{i,j}v) = \begin{cases} 2nm+nj-n+i+2, & \text{if } i \text{ odd, } 1 \le i \le n-1,\ 1 \le j \le m \\ i+n(j+m-1)+1, & \text{if } i \text{ even, } 2 \le i \le n,\ 1 \le j \le m \end{cases}$$

$$w(y_{i,j}v) = \begin{cases} i+n(j+m-1)+1, & \text{if } i \text{ odd, } 1 \le i \le n-1,\ 1 \le j \le m \\ 2nm+nj-n+i+2, & \text{if } i \text{ even, } 2 \le i \le n,\ 1 \le j \le m \end{cases}$$

$$w(x_{i,j}y_{i,j}) = nm+2nj-2n+2i+1, \qquad 1 \le i \le n,\ 1 \le j \le m$$

$$w(x_{i,j}x_{i+1,j}) = 3nm+2nj-n+2i+4, \qquad 1 \le i \le n-1,\ 1 \le j \le m$$

$$w(x_{n,j}x_{1,j}) = 2nj+3nm-n+4, \qquad 1 \le j \le m$$

$$w(y_{i,j}y_{i+1,j}) = 3nm+2nj-n+2i+4, \qquad 1 \le i \le n-1,\ 1 \le j \le m$$

$$w(y_{n,j}y_{1,j}) = nm+2nj-2n+i+2, \qquad 1 \le j \le m$$

The edge weight above will induce the rainbow antimagic coloring of graph, such that the number of different edge weight as follows:

$W_1 = w(x_{i,j}v) = W_{1a} \cup W_{1b}$, a for i odd $(1 \le i < n-1)$, b for i even $(2 \le i \le n)$

$W_{1a} = \{2nm+3, 2nm+5, 2mn+7, \ldots, 2nm+mn+1\}$.

$$U_{|W_{1a}|} = a + (|W_{1a}|-1)b$$

$$2nm+mn+1 = 2nm+3+(|W_{1a}|-1)\cdot 2$$
$$|W_{1a}| = \frac{mn}{2}.$$

$W_{1b} = \{nm+3,\ nm+5, nm+7,\ \ldots,\ 2nm+1\}$.

$$U_{|W_{1b}|} = a + (|W_{1b}|-1)b$$

$$2nm+1 = nm+3+(|W_{1b}|-1)\cdot 2$$
$$|W_{1b}| = \frac{mn}{2}.$$

$|W_1| = |W_{1a}| + |W_{1b}| = mn$

$W_2 = w(x_{i,j}v) = W_{2a} \cup W_{2b}$, a for i odd $(1 \le i < n-1)$, b for i even $(2 \le i \le n)$

$W_{2a} = \{nm+2,\ nm+4,\ nm+6,\ \ldots,\ 2mn\}$.

$$U_{|W_{2a}|} = a + (|W_{2a}|-1)b$$

$$2mn = nm+2+(|W_{2a}|-1)\cdot 2$$
$$|W_{2a}| = \frac{mn}{2}.$$

$W_{2b} = \{2nm + 4,\ 2nm + 6,\ 2nm + 8,\ \ldots,\ 2nm + mn + 2\}.$

$$U_{|W_{2b}|} = a + (|W_{2b}| - 1)b$$

$$2nm + mn + 2 = 2nm + 4 + (|W_{2b}| - 1) \cdot 2$$

$$|W_{2b}| = \frac{mn}{2}.$$

$|W_2| = |W_{2a}| + |W_{2b}| = mn$

We aim to prove that the intersection of the sets W_1 and W_2 is empty, i.e., $W_1 \cap W_2 = \emptyset$. For W_1, the minimum value occurs when $i = 1$ and $j = 1$, giving $w_1(1, 1) = 2nm + n(1) - n + 1 + 2 = 2nm + 3$. The maximum value occurs when $i = n-1$ and $j = m$, yielding $w_1(n-1, m) = 2nm+n(m)-n+(n-1)+2 = 3nm + 1$. Thus, the range of W_1 for odd i is $W_1 = [\,2nm + 3,\ 3nm + 1\,]$. For even i,the minimum value occurs when $i = 2$ and $j = 1$, giving $w_1(2, 1) = 2 + n(1 + m - 1) + 1 = nm + 3$. The maximum value occurs when $i = n$ and $j = m$, giving $w_1(n, m) = n + n(m + m - 1) + 1 = 2nm - n + 1$. Thus, for even i, the range of W_1 is $W_1 = [\,nm + 3,\ 2nm - n + 1\,]$.

For W_2, the weight function for odd i the minimum value occurs when $i = 1$ and $j = 1$, giving $w_2(1, 1) = 1 + n(1 + m - 1) + 1 = nm + 2$. The maximum value occurs when $i = n - 1$ and $j = m$, giving $w_2(n - 1, m) = (n - 1) + n(m + m - 1) + 1 = 2nm - n + 1$. Thus, the range of W_2 for odd i is $W_2 = [\,nm + 2, 2nm - n + 1\,]$. For even i, the minimum value occurs when $i = 2$ and $j = 1$, giving $w_2(2, 1) = 2nm + n(1) - n + 2 + 2 = 2nm + 4$. The maximum value occurs when $i = n$ and $j = m$, giving $w_2(n, m) = 2nm + n(m) - n + n + 2 = 3nm + n + 2$. Thus, for even i, the range of W_2 is $W_2 = [\,2nm + 4,\ 3nm + n + 2\,]$.

To do so, we analyze the structure of each set and show that no element in W_1 can also belong to W_2. When i is odd, to check for any overlap, we equate the two expressions for W_1 and W_2 is $2nm + nj - n + i + 2 = i + n(j + m - 1) + 1$. Simplifying both sides, we obtain $2nm + nj - n + i + 2 = i + nj + nm - n + 1$ Rearranging terms, we get $2nm + 2 \neq nm + 1$. This contradiction shows that there is no solution for i odd, and hence, no element in W_1 when i is odd can also belong to W_2.

When i is even, we check for any overlap by equating the two expressions for W_1 and W_2 is $i + n(j + m - 1) + 1 = 2nm + nj - n + i + 2$. Simplifying both sides, we obtain $n(j + m - 1) + 1 = 2nm + nj - n + 2$ Rearranging terms, we get $n(j + m - 1) + 1 = nj + 2nm - n + 2$. This leads to a contradiction, as the left-hand side and right-hand side are not equal for any values of i and j.

Since there is no solution in either case for i odd or even, we conclude that there are no elements common to both W_1 and W_2. Therefore, the intersection of W_1 and W_2 is the empty set:

$$W_1 \cap W_2 = \emptyset$$

We are tasked with proving that $W_3 \subseteq W_1 \cup W_2$. To do so, we begin by analyzing the weight function for W_3, which is given by The minimum value of $w_3(i, j)$ occurs when $i = 1$ and $j = 1$, giving $w_3(1, 1) = nm+2n(1)-2n+2(1)+1 = nm+3$. The maximum value occurs when $i = n$ and $j = m$, giving $w_3(n, m) = nm + 2n(m) - 2n + 2n + 1 = 3nm + 1$. Thus, the range of W_3 is $W_3 \subseteq [\,nm+3,\ 3nm+1\,] \cap \mathbb{Z}$.

Next, we examine the ranges of W_1 and W_2 to show that the range of W_3 lies within the union of these two sets. The range of W_3 is $[\,nm + 3,\ 3nm + 1\,]$, which is fully contained within the union of the ranges of W_1 and W_2, as the lower bound of W_3 (which is $nm + 3$) is covered by W_1 and the upper bound of W_3 (which is $3nm + 1$) is covered by both W_1 and W_2. Thus, we conclude that

$$W_3 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_4 \subseteq W_1 \cup W_2$. To analyze the range of W_4, we first find the minimum and maximum values of the weight function. The minimum value occurs when both $i = 1$ and $j = 1$. Substituting these values into the weight function, we get $w_4(1, 1) = 3nm + 2n(1) - n + 2(1) + 4 = 3nm + 2n - n + 2 + 4 = 3nm + n + 6$. Thus, the minimum value of W_4 is $W_4^{\min} = 3nm + n + 6$. On the other hand, the maximum value occurs when $i = n - 1$ and $j = m$, giving $w_4(n - 1, m) = 3nm + 2n(m) - n + 2(n - 1) + 4 = 3nm + 2nm - n + 2n - 2 + 4 = 5nm + 2$. Thus, the maximum value of W_4 is $W_4^{\max} = 5nm + 2$. This means the range of W_4 is $W_4 \subseteq [\,3nm + n + 6,\ 5nm + 2\,] \cap \mathbb{Z}$.

Next, we compare the range of W_4, which is $[\,3nm + n + 6,\ 5nm + 2\,]$, with the ranges of W_1 and W_2. The range of W_1 is $[\,2nm + 3,\ 3nm + 1\,]$ for odd i, and $[\,nm + 3,\ 2nm - n + 1\,]$ for even i. The range of W_2 is $[\,nm + 2,\ 2nm - n + 1\,]$ for odd i, and $[\,2nm + 4,\ 3nm + n + 2\,]$ for even i. By comparing these ranges, we observe that the interval $[\,3nm + n + 6,\ 5nm + 2\,]$ is fully contained within the union of the ranges of W_1 and W_2.

Therefore, we conclude that

$$W_4 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_5 \subseteq W_1 \cup W_2$. To find the range of W_5, we begin by determining the minimum and maximum values of $w_5(j)$. The minimum value occurs when $j = 1$, giving $w_5(1) = 2n(1)+3nm-n+4 = 2n+3nm-n+4 = 3nm+n+4$. Thus, the minimum value of W_5 is $W_5^{\min} = 3nm+n+4$. On the other hand, the maximum value occurs when $j = m$, giving $w_5(m) = 2n(m) + 3nm - n + 4 = 2nm + 3nm - n + 4 = 5nm - n + 4$. Thus, the range of W_5 is $W_5 \subseteq [\,3nm + n + 4,\, 5nm - n + 4\,] \cap \mathbb{Z}$. Next, we compare the range of W_5, which is $[\,3nm + n + 4,\, 5nm - n + 4\,]$, with the ranges of W_1 and W_2. The range of W_1 is $[\,2nm + 3,\, 3nm + 1\,]$ for odd i, and $[\,nm + 3,\, 2nm - n + 1\,]$ for even i. The range of W_2 is $[\,nm + 2,\, 2nm - n + 1\,]$ for odd i, and $[\,2nm + 4,\, 3nm + n + 2\,]$ for even i. By comparing these ranges, we observe that the interval $[\,3nm + n + 4,\, 5nm - n + 4\,]$ is fully contained within the union of the ranges of W_1 and W_2. Therefore, we conclude that

$$W_5 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_6 \subseteq W_1 \cup W_2$. To analyze the range of W_6, we first find the minimum and maximum values of the weight function. The minimum value occurs when $i = 1$ and $j = 1$, giving $w_6(1, 1) = 3nm + 2n(1) - n + 2(1) + 4 = 3nm + 2n - n + 2 + 4 = 3nm + n + 6$. Thus, the minimum value of W_6 is $W_6^{\min} = 3nm+n+6$. On the other hand, the maximum value occurs when $i = n-1$ and $j = m$, giving $w_6(n-1, m) = 3nm+2n(m)-n+2(n-1)+4 = 3nm+2nm-n+2n-2+4 = 5nm+2$. Thus, the maximum value of W_6 is $W_6^{\max} = 5nm+2$. Hence, the range of W_6 is $W_6 \subseteq [\,3nm+n+6,\, 5nm+2\,] \cap \mathbb{Z}$.

Next, we compare the range of W_6, which is $[\,3nm + n + 6,\, 5nm + 2\,]$, with the ranges of W_1 and W_2. The range of W_1 is $[\,2nm + 3,\, 3nm + 1\,]$ for odd i, and $[\,nm + 3,\, 2nm - n + 1\,]$ for even i. The range of W_2 is $[\,nm + 2,\, 2nm - n + 1\,]$ for odd i, and $[\,2nm + 4,\, 3nm + n + 2\,]$ for even i. By comparing these ranges, we observe that the interval $[\,3nm + n + 6,\, 5nm + 2\,]$ is fully contained within the union of the ranges of W_1 and W_2. Therefore, we conclude that

$$W_6 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_7 \subseteq W_1 \cup W_2$. To find the range of W_7, we first determine the minimum and maximum values of the weight function. The minimum value occurs when $i = 1$ and $j = 1$, giving $w_7(1, 1) = nm + 2n(1) - 2n + 1 + 2 = nm + 2n - 2n + 1 + 2 = nm + 3$. Thus, the minimum value of W_7 is $W_7^{\min} = nm + 3$. On the other hand, the maximum value occurs when $i = n$ and $j = m$, giving $w_7(n, m) = nm + 2n(m) - 2n + n + 2 = nm + 2nm - 2n + n + 2 = 3nm - n + 2$.

Thus, the maximum value of W_7 is $W_7^{\max} = 3nm - n + 2$. Hence, the range of W_7 is $W_7 \subseteq [\,nm + 3,\, 3nm - n + 2\,] \cap \mathbb{Z}$.

Next, we compare the range of W_7, which is $[\,nm + 3,\, 3nm - n + 2\,]$, with the ranges of W_1 and W_2. The range of W_1 is $[\,2nm+3,\, 3nm+1\,]$ for odd i, and $[\,nm+3,\, 2nm-n+1\,]$ for even i. The range of W_2 is $[\,nm + 2,\, 2nm - n + 1\,]$ for odd i, and $[\,2nm + 4,\, 3nm + n + 2\,]$ for even i. By comparing these ranges, we observe that the interval $[\,nm + 3,\, 3nm - n + 2\,]$ is fully contained within the union of the ranges of W_1 and W_2. Therefore, we conclude that

$$W_7 \subseteq W_1 \cup W_2.$$

Therefore, the number of distinct edge weights is $|W_1 \cup W_2| = |W_1| + |W_2| = mn + mn = 2mn$.This implies that $rac(Amal(Le_n, v, m)) \leq 2mn$. Since $rac(Amal(Le_n, v, m)) \geq 2mn$ and $rac(Amal(Le_n, v, m)) \leq 2mn$, we conclude that $rac(Amal(Le_n, v, m)) = 2mn$.

For any two vertices, there exists a rainbow path, as presented in Table 2.

TABLE 2. $y - z$ Rainbow path in amalgamation of lemon graph $Amal(Le_n, v, m)$

y	**z**	**rainbow path**
$x_{i,j}$	v	$x_{i,j}, v$
$y_{i,j}$	v	$y_{i,j}, v$
$x_{i,j}$	$y_{i,j}$	$x_{i,j}, v, y_{i,j}$
$x_{i,j}$	$x_{k,l}$	$x_{i,j}, v, x_{k,l}$
$y_{i,j}$	$y_{k,}$	$y_{i,j}, v, y_{k,l}$
$x_{i,j}$	$y_{k,l}$	$x_{i,j}, v, y_{k,l}$

It can be easily observed from the table above that for any two vertices, there exists a rainbow path between them, which satisfies the condition for a valid rainbow antimagic coloring. □

As an example can be viewed in Figure 2.

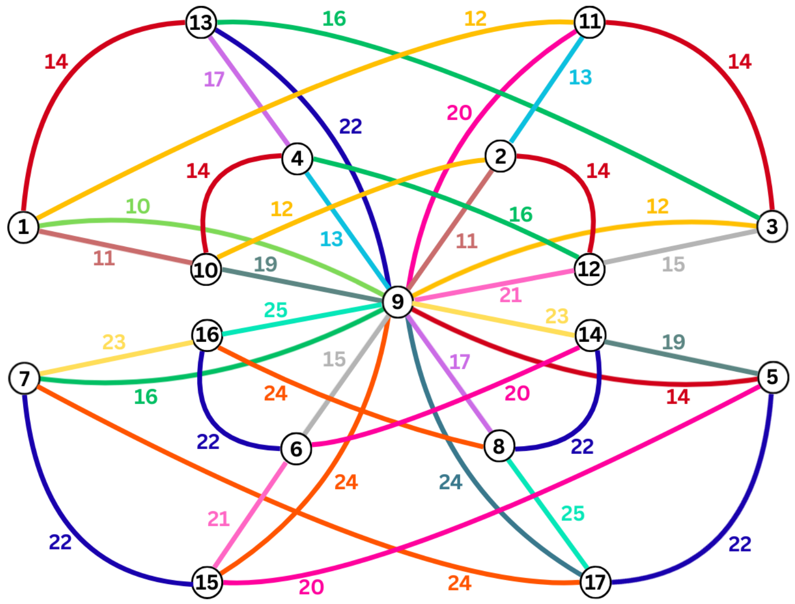

FIGURE 2. A rainbow antimagic coloring of $rac(Amal(Le_4, v, 2))$

Theorem 3. *Let $Amal(DW_n, v, m)$ be a amalgamation of double wheel graph, with the central vertex as the linking vertex. For every positive integer $n \geq 3$, we have $rac(Amal(DW_n$ $, v, m)) = 2mn$.*

Proof. $Amal(DW_n; v, m)$ is amalgamation of double wheel graph with the central vertex as the linking vertex with vertex set $V(Amal(DW_n, v, m)) = \{x_{i,j}, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{y_{i,j}, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{v\}$ and edge set $E(Amal(DW_n, v, m)) = \{x_{i,j}v, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{y_{i,j}v, 1 \leq i \leq n \ \& \ 1 \leq i \leq m\} \cup \{x_{i,j}x_{i+1,j}, 1 \leq i \leq n-1 \ \& \ 1 \leq i \leq m\} \cup \{x_{i=n,j}x_{1,j}, 1 \leq i \leq m\} \cup \{y_{i,j}y_{i+1,j}, 1 \leq i \leq n-1 \ \& \ 1 \leq i \leq m\} \cup \{y_{i=n,j}y_{1,j}, 1 \leq i \leq m\}$. The order and size of $Amal(DW_n, v, m)$ are $2mn + 1$ and $4mn$, respectively. The maximum degree of $Amal(DW_n, v, m)$ is $2mn$.

To prove that $rac(Amal(DW_n, v, m)) = 2mn$, we must establish both the lower bound, $rac(Amal(DW_n, v, m)) \geq 2mn$, and the upper bound, $rac(Amal(DW_n, v, m)) \leq 2mn$. Based on Lemma 1, we have $rac(Amal(DW_n, v, m)) \geq max\{rc(Amal(DW_n, v, m))\Delta(Amal, (DW_n, v, m))\} = max\{4m, 2mn\} = 2mn$. Next, we show the upper bound $rac(Amal(DW_n, v, m)) \leq 2mn$ using the following bijective functions.

Case 1. for n =odd

$f(v) = m(n-1) + 1$

$$f(x_{i,j}) = \begin{cases} \dfrac{i + jn - j - n + 2}{2}, & \text{if } i \text{ odd, for } 1 \leq i \leq n-2,\ 1 \leq j \leq m \\ 2m + 2 + \frac{3m(n-1) - i + j + n - jn - 1}{2}, & \text{if } i \text{ even, for } 2 \leq i \leq n-1,\ 1 \leq j \leq m \\ m(n-1) + m + j + 1, & \text{for } i = n,\ 1 \leq j \leq m \end{cases}$$

$$f(y_{i,j}) = \begin{cases} \dfrac{mn + jn - m - j - n + i + 2}{2}, & \text{if } i \text{ odd, for } 1 \leq i \leq n-2,\ 1 \leq j \leq m \\ 2m + 3 + \frac{4m(n-1) + j(1-n) + n - 3 - i}{2}, & \text{if } i \text{ even, for } 2 \leq i \leq n-1,\ 1 \leq j \leq m \\ m(n-1) + j + 1, \quad 1 \leq j \leq m \end{cases}$$

Based on a vertex function, the edge weight is as follows:

$$w(x_{i,j}v) = \begin{cases} \dfrac{i+jn-j-n+2}{2} + m(n-1)+1, & \text{if } i \text{ odd, } 1 \le i \le n-2,\ 1 \le j \le m \\ 2m+3+\frac{3m(n-1)-i+j+n-jn-1}{2} + m(n-1), & \text{if } i \text{ even, } 2 \le i \le n-1,\ 1 \le j \le m \\ 2m(n-1)+m+j+2, \quad 1 \le j \le m \end{cases}$$

$$w(y_{i,j}v) = \begin{cases} \frac{(m+j)n-n-m-j+i+2}{2} + m(n-1)+1, & \text{if } i \text{ odd, } 1 \le i \le n-2,\ 1 \le j \le m \\ 2m+4+\frac{4m(n-1)+j(1-n)+n-3-i}{2} + m(n-1), & \text{if } i \text{ even, } 2 \le i \le n-1,\ 1 \le j \le m \\ 2m(n-1)+j+2, \quad 1 \le j \le m \end{cases}$$

$$w(x_{i,j}x_{i+1,j}) = \begin{cases} 2m+2+\dfrac{3m(n-1)}{2}, & \text{if } i \text{ odd, } 1 \le i \le n-2,\ 1 \le j \le m \\ 2m+2+\frac{3m(n-1)+2j(n-1)+3-n}{2}, & \text{if } i \text{ even, } 1 \le i \le n-1,\ 1 \le j \le m \end{cases}$$

$$w(x_{n,j}x_{1,j}) = mn+j+1+\frac{3+jn-j-n}{2}, \qquad 1 \le j \le m$$

$$w(y_{i,j}y_{i+1,j}) = \begin{cases} \dfrac{5mn+6}{2}, & \text{if } i \text{ odd, } 1 \le i \le n-2,\ 1 \le j \le m \\ \dfrac{5mn-m+6}{2}, & \text{if } i \text{ even, } 1 \le i \le n-1,\ 1 \le j \le m \end{cases}$$

$$w(y_{n,j}y_{1,j}) = \frac{3mn+jn-3m+j-n+5}{2}, \qquad 1 \le j \le m$$

The edge weight above will induce the rainbow antimagic coloring of graph, such that the number of different edge weight as follows:
$W_1 = w(x_{i,j}v) = W_{1a} \cup W_{1b} \cup W_{1c}$, a for i odd $(1 \le i < n-2)$, b for i even $(2 \le i \le n-1)$, c for $i = n$

$$W_{1a} = \{\, m(n-1)+2,\ m(n-1)+3,\ \tfrac{n+1}{2} + m(n-1)+1, \ldots, \tfrac{3m(n-1)}{2}\}$$

$$U_{|W_{1a}|} = a + (|W_{1a}|-1)b$$
$$\frac{3m(n-1)}{2}\} = m(n-1)+2+(|W_{1a}|-1)\cdot 1$$
$$|W_{1a}| = \tfrac{(n-1)m}{2}$$

$$W_{1b} = \{\, 2m+3+\tfrac{3m(n-1)+n-4}{2} + m(n-1), \ldots, 2m+3+\tfrac{3m(n-1)+n-1+m+n-mn-1}{2} + m(n-1)\,\}$$

$$U_{|W_{1b}|} = a + (|W_{1b}|-1)b$$
$$2m+3+\frac{3m(n-1)+n-1+m+n-mn-1}{2} + m(n-1)$$
$$= 2m+3+\frac{3m(n-1)+n-4}{2} + m(n-1) + (|W_{1b}|-1)\cdot -1$$
$$|W_{1b}| = \tfrac{(n-1)m}{2}$$

$$W_{1c} = \{\, 2m(n-1)+m+3,\ 2m(n-1)+m+4, \ldots, 2m(n-1)+2m+2\,\}$$

$$U_{|W_{1c}|} = a + (|W_{1c}|-1)b$$
$$2m(n-1)+2m+2 = 2m(n-1)+m+3+(|W_{1c}|-1)\cdot 1$$
$$|W_{1c}| = m$$
$$W_1 = W_{1a} \cup W_{1b} \cup W_{1c}, |W_1| = |W_{1a}|+|W_{1b}|+|W_{1c}| = \tfrac{(n-1)m}{2} + \tfrac{(n-1)m}{2} + m = mn$$

$W_2 = w(y_{i,j}v) = W_{2a} \cup W_{2c} \cup W_{2b}$, a for i odd $(1 \le i < n)$, b for i even $(2 \le i \le n)$, c for $i = n$,

$$W_{2a} = \{\, \tfrac{3m(n-1)}{2} + 2,\ \tfrac{3m(n-1)}{2} + 3, \ldots,\ 2m(n-1)+1\,\}$$

$$U_{|W_{2a}|} = a + (|W_{2a}| - 1)b$$
$$2m(n-1) + 1 = \frac{3m(n-1)}{2} + 2 + (|W_{2a}| - 1) \cdot 1$$
$$|W_{2a}| = \tfrac{(n-1)m}{2}$$
$$W_{2b} = \{\, 2m + 2 + 3m(n-1),\ 2m + 1 + 3m(n-1), \dots,\ 2m + 4 + 2m(n-1) \,\}$$

$$U_{|W_{2b}|} = a + (|W_{2b}| - 1)b$$
$$2m + 4 + 2m(n-1) = 2m + 2 + 3m(n-1) + (|W_{2b}| - 1) \cdot -1$$
$$|W_{2b}| = \tfrac{(n-1)m}{2}$$
$$W_{2c} = \{\, 2m(n-1) + 3,\ 2m(n-1) + 4,\ 2m(n-1) + 5, \dots,\ 2m(n-1) + m + 2 \,\}$$

$$U_{|W_{2c}|} = a + (|W_{2c}| - 1)b$$
$$2m(n-1) + m + 2 = 2m(n-1) + 3 + (|W_{2c}| - 1) \cdot 1$$
$$|W_{2c}| = m$$
$$W_2 = W_{2a} \cup W_{2b} \cup W_{2c}, |W_2| = |W_{2a}| + |W_{2b}| + |W_{2c}| = \tfrac{(n-1)m}{2} + \tfrac{(n-1)m}{2} + m = mn.$$

We aim to prove that the intersection of the sets W_1 and W_2 is empty, i.e., $W_1 \cap W_2 = \emptyset$. To achieve this, we analyze the weight functions $W_1(i,j)$ and $W_2(i,j)$ under different conditions for i, considering the cases when i is odd, even, and when $i = n$. For odd values of i where $1 \le i \le n-2$, the weight functions are $W_1(i,j) = \frac{i+jn-j-n+2}{2} + m(n-1) + 1$ and $W_2(i,j) = \frac{(m+j)n-n-m-j+i+2}{2} + m(n-1) + 1$. Upon simplifying both expressions, we observe that the terms $m(n-1)+1$ are common to both functions. Therefore, we focus on $W_1(i,j) = \frac{i+jn-j-n+2}{2}$, $W_2(i,j) = \frac{(m+j)n-n-m-j+i+2}{2}$. The expressions differ due to the presence of the terms $-m-j$ in W_2, which are absent in W_1. Consequently, $W_1(i,j) \ne W_2(i,j)$ for any odd value of i.

For even values of i, where $2 \le i \le n-1$, the weight functions are $W_1(i,j) = 2m+3+\frac{3m(n-1)-i+j+n-jn-1}{2} + m(n-1)$ and $W_2(i,j) = 2m+4+\frac{4m(n-1)+j(1-n)+n-3-i}{2} + m(n-1)$. Upon simplifying both expressions, we note that the terms involving $2m + m(n-1)$ are present in both functions, but there is a difference in the constants. Specifically, $2m+3$ in W_1 and $2m+4$ in W_2 differ by 1. Additionally, the fractional parts differ due to the presence of the terms involving m and j in W_2, further ensuring that $W_1(i,j) \ne W_2(i,j)$ for any even value of i. Finally, for $i = n$, the weight functions are $W_1(i,j) = 2m(n-1) + m + j + 2$ and $W_2(i,j) = 2m(n-1) + j + 2$. The only difference between these two expressions is the term m in W_1, which does not appear in W_2. Therefore, $W_1(i,j) \ne W_2(i,j)$ for any value of j when $i = n$.

From the analysis of all cases odd i, even i, and $i = n$, we have shown that there are no values of (i,j) for which $W_1(i,j) = W_2(i,j)$. Therefore, the intersection of the sets W_1 and W_2 is empty, and we conclude that

$$W_1 \cap W_2 = \emptyset.$$

We are tasked with proving that $W_3 \subseteq W_1 \cup W_2$. To do this, we first calculate the minimum and maximum values for W_3 and then compare these values with the ranges of W_1 and W_2. For the first case, where i is odd and $1 \le i \le n-2$, $1 \le j \le m$, the minimum value occurs when $i = 1$ and $j = 1$, giving $2m + 2 + \frac{3m(n-1)}{2}$. The maximum value occurs when $i = n-2$ and $j = m$, giving $2m + 2 + \frac{3m(n-1)+2m(n-1)+3-n}{2}$. For the second case, when i is even and $1 \le i \le n-1$, $1 \le j \le m$, the minimum value occurs when $i = 2$ and $j = 1$, giving $2m + 2 + \frac{3m(n-1)+2(n-1)+3-n}{2}$. The maximum value occurs when $i = n-1$ and $j = m$, giving $2m + 2 + \frac{3m(n-1)+2m(n-1)+3-n}{2}$. Now, we compare these values with the ranges of W_1 and W_2. The range of W_1 is $[3, 3nm+2]$, and the range of W_2 is $[3, 2nm+2]$. By comparing the minimum and maximum values of W_3 with these ranges, we observe that the values of W_3 fall within the interval covered by W_1 and W_2. Therefore, we conclude that

$$W_3 \subseteq W_1 \cup W_2$$

We are tasked with proving that $W_4 \subseteq W_1 \cup W_2$. To do this, we first calculate the minimum and maximum values for W_4 and then compare these values with the ranges of W_1 and W_2. For the first case, the minimum value of W_4 occurs when $j = 1$, giving $mn + 1 + \frac{3+n-1}{2}$. The maximum value occurs when $j = m$, giving $mn + m + 1 + \frac{3+mn-m-n}{2}$. Now, we compare these values with the ranges of W_1 and W_2. The range of W_1 is $[3, 3nm+2]$, and the range of W_2 is $[3, 2nm+2]$. By comparing the minimum and maximum values of W_4 with these ranges, we observe that the values of W_4 fall within the interval covered by W_1 and W_2. Therefore, we conclude that $W_4 \subseteq W_1 \cup W_2$. Therefore, we conclude that

$$W_4 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_5 \subseteq W_1 \cup W_2$. To do this, we first calculate the minimum and maximum values for W_5 and then compare these values with the ranges of W_1 and W_2. For the case when i is odd, the minimum value of W_5 occurs when $i = 1$ and $j = 1$, giving $\frac{5mn+6}{2}$. The maximum value occurs when $i = n - 2$ and $j = m$, giving $\frac{5mn+6}{2}$. For the case when i is even, the minimum value of W_5 occurs when $i = 2$ and $j = 1$, giving $\frac{5mn-m+6}{2}$. The maximum value occurs when $i = n - 1$ and $j = m$, giving $\frac{5mn-m+6}{2}$. Now, we compare these values with the ranges of W_1 and W_2. The range of W_1 is $[3, 3nm + 2]$, and the range of W_2 is $[3, 2nm + 2]$. By comparing the minimum and maximum values of W_5 with these ranges, we observe that the values of W_5 lie within the interval covered by W_1 and W_2. Therefore, we conclude that

$$W_5 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_6 \subseteq W_1 \cup W_2$. To do this, we first calculate the minimum and maximum values for W_6 and then compare these values with the ranges of W_1 and W_2. For the first case, the minimum value of W_6 occurs when $j = 1$, giving $\frac{3mn+n-3m+1-n+5}{2} = \frac{3mn-3m+6}{2}$. The maximum value occurs when $j = m$, giving $\frac{3mn+mn-3m+m-n+5}{2} = \frac{4mn-3m-n+5}{2}$. Now, we compare these values with the ranges of W_1 and W_2. The range of W_1 is $[3, 3nm + 2]$, and the range of W_2 is $[3, 2nm + 2]$. By comparing the minimum and maximum values of W_6 with these ranges, we observe that the values of W_6 lie within the interval covered by W_1 and W_2. Therefore, we conclude that

$$W_6 \subseteq W_1 \cup W_2.$$

Therefore, the number of distinct edge weights is $|W_1| + |W_2| = mn + mn = 2mn$. This implies that $rac(Amal(DW_n, v, m)) \leq 2mn$. Since $rac(Amal(DW_n, v, m)) \geq 2mn$ and $rac(Amal(DW_n, v, m)) \leq 2mn$, we conclude that $rac(Amal(DW_n, v, m)) = 2mn$.

Case 2. for $n =$ even

$$f(v) = mn + 1$$

$$f(x_{i,j}) = \begin{cases} \dfrac{i+1}{2} + \dfrac{jn}{2} - \dfrac{n}{2}, & \text{if } i \text{ odd, for } 1 \leq i \leq n-1,\ 1 \leq j \leq m \\ mn - \dfrac{n}{2} + \dfrac{jn}{2} + \dfrac{i}{2} + 1, & \text{if } i \text{ even, for } 2 \leq i \leq n,\ 1 \leq j \leq m \end{cases}$$

$$f(y_{i,j}) = \begin{cases} \dfrac{mn}{2} + \dfrac{i+1}{2} + \dfrac{jn}{2} - \dfrac{n}{2}, & \text{if } i \text{ odd, for } 1 \leq i \leq n-1,\ 1 \leq j \leq m \\ \dfrac{3mn}{2} + 1 + \dfrac{i}{2} + \dfrac{n(j-1)}{2}, & \text{if } i \text{ even, for } 2 \leq i \leq n,\ 1 \leq j \leq m \end{cases}$$

Based on a vertex function, the edge weight is as follows:

$$w(x_{i,j}v) = \begin{cases} \dfrac{i+1}{2} + \dfrac{jn}{2} - \dfrac{n}{2} + mn + 1, & \text{if } i \text{ odd, } 1 \leq i \leq n-1,\ 1 \leq j \leq m \\ 2mn + \dfrac{n(j-1)}{2} + \dfrac{i}{2} + 2, & \text{if } i \text{ even, } 2 \leq i \leq n,\ 1 \leq j \leq m \end{cases}$$

$$w(y_{i,j}v) = \begin{cases} \dfrac{mn}{2} + \dfrac{i+1}{2} + \dfrac{jn}{2} - \dfrac{n}{2} + mn + 1, & \text{if } i \text{ odd, } 1 \leq i \leq n-1,\ 1 \leq j \leq m \\ \dfrac{5mn}{2} + 2 + \dfrac{i}{2} + \dfrac{n(j-1)}{2}, & \text{if } i \text{ even, } 2 \leq i \leq n,\ 1 \leq j \leq m \end{cases}$$

$$w(x_{i,j}x_{i+1,j}) = \begin{cases} i + jn + mn - n + 2, & \text{if } i \text{ odd, } 1 \leq i \leq n-1,\ 1 \leq j \leq m \\ mn + jn + i - n + 2, & \text{if } i \text{ even, } 2 \leq i \leq n-2,\ 1 \leq j \leq m \end{cases}$$

$$w(x_{n,j}x_{1,j}) = mn + jn - \frac{n}{2} + 2, \qquad 1 \leq j \leq m$$

$$w(y_{i,j}y_{i+1,j}) = \begin{cases} 2mn + i + 2 + n(j-1), & \text{if } i \text{ odd}, \ 1 \le i \le n-1, \ 1 \le j \le m \\ 2mn + i + 2 + n(j-1), & \text{if } i \text{ even}, \ 2 \le i \le n-2, \ 1 \le j \le m \end{cases}$$

$$w(y_{n,j}y_{1,j}) = 2mn + 2 + \frac{n(2j-1)}{2}, \qquad 1 \le j \le m$$

The edge weight above will induce the rainbow antimagic coloring of graph, such that the number of different edge weight as follows:
$W_1 = w(x_{i,j}v) = W_{1a} \cup W_{1b}$, a for i odd $(1 \le i < n-1)$, b for i even $(2 \le i \le n)$

$$W_{1a} = \{mn+2, mn+3, mn+4, \ldots, \frac{3mn}{2}+1\}.$$

$$U_{|W_{1a}|} = a + (|W_{1a}|-1)b$$

$$\frac{3mn}{2} + 1 = mn + 2 + (|W_{1a}|-1) \cdot 1$$
$$|W_{1a}| = \frac{mn}{2}.$$

$$W_{1b} = \{2mn+3, \ 2mn+4, 2mn+5, \ \ldots, \ 2mn + \frac{mn}{2} + 2\}.$$

$$U_{|W_{1b}|} = a + (|W_{1b}|-1)b$$

$$2mn + \frac{mn}{2} + 2 = 2mn + 3 + (|W_{1b}|-1) \cdot 1$$
$$|W_{1b}| = \frac{mn}{2}.$$

$|W_1| = |W_{1a}| + |W_{1b}| = mn$

$W_2 = w(x_{i,j}v) = W_{2a} \cup W_{2b}$, a for i odd $(1 \le i < n-1)$, b for i even $(2 \le i \le n)$

$$W_{2a} = \{\frac{3mn}{2}+2, \ \frac{3mn}{2}+3, \ \frac{3mn}{2}+4, \ \ldots, \ 2mn+1\}.$$

$$U_{|W_{2a}|} = a + (|W_{2a}|-1)b$$

$$2mn + 1 = \frac{3mn}{2} + 2 + (|W_{2a}|-1) \cdot 1$$
$$|W_{2a}| = \frac{mn}{2}.$$

$$W_{2b} = \{\frac{5mn}{2}+3, \ \frac{5mn}{2}+4, \ \frac{5mn}{2}+5, \ \ldots, \ 3mn+2\}.$$

$$U_{|W_{2b}|} = a + (|W_{2b}|-1)b$$

$$3mn + 2 = \frac{5mn}{2} + 3 + (|W_{2b}|-1) \cdot 1$$
$$|W_{2b}| = \frac{mn}{2}.$$

$|W_2| = |W_{2a}| + |W_{2b}| = mn$

We now show that the intersection of the sets W_1 and W_2 is empty. To demonstrate that the intersection $W_1 \cap W_2 = \emptyset$, we analyze the minimum and maximum values of both sets. For W_1, consider the case when i is odd, the minimum value occurs when $i = 1$ and $j = 1$, giving $\frac{1+1}{2} + \frac{n(1)}{2} - \frac{n}{2} + mn + 1 = mn + 2$. The maximum value occurs when $i = n-1$ and $j = m$, giving $\frac{n}{2} + \frac{n(m)}{2} - \frac{n}{2} + mn + 1 = mn + 2$. For the case when i is even, the minimum value occurs when $i = 2$ and $j = 1$, giving $2mn + \frac{n(0)}{2} + \frac{2}{2} + 2 = 2mn + 3$.

The maximum value occurs when $i = n$ and $j = m$, giving $2mn + \dfrac{n(m-1)}{2} + \dfrac{n}{2} + 2 = 2mn + nm + 2$. For W_2, consider the case when i is odd, the minimum value occurs when $i = 1$ and $j = 1$, giving $\dfrac{mn}{2} + \dfrac{1+1}{2} + \dfrac{n(1)}{2} - \dfrac{n}{2} + mn + 1 = mn + 2$. The maximum value occurs when $i = n-1$ and $j = m$, giving $\dfrac{mn}{2} + \dfrac{n-1+1}{2} + \dfrac{n(m)}{2} - \dfrac{n}{2} + mn + 1 = mn + 2$. For the case when i is even, the minimum value occurs when $i = 2$ and $j = 1$, giving $\dfrac{5mn}{2} + 2 + \dfrac{2}{2} + \dfrac{n(0)}{2} = \dfrac{5mn}{2} + 3$. The maximum value occurs when $i = n$ and $j = m$, giving $\dfrac{5mn}{2} + 2 + \dfrac{n}{2} + \dfrac{n(m-1)}{2} = \dfrac{5mn}{2} + 2 + \dfrac{n(m)}{2}$.

Now, we compare the ranges of W_1 and W_2. The range of W_1 for i odd is $[mn+2, mn+2]$, which coincides with the range of W_2 for i odd. However, the ranges for even i are distinct. Specifically, for even i, W_1 begins at $2mn+3$, while W_2 starts at $\dfrac{5mn}{2} + 3$, which is strictly greater than $2mn+3$. This indicates that there is no overlap between the two ranges.

Since there is no overlap in the ranges of W_1 and W_2, we conclude that the intersection $W_1 \cap W_2$ is indeed empty. Therefore, we have shown that

$$W_1 \cap W_2 = \emptyset.$$

We are tasked with proving that $W_3 \subseteq W_1 \cup W_2$. First, we compute the minimum and maximum values of W_3 for each case. When i is odd, the minimum value occurs when $i = 1$ and $j = 1$, giving $1 + 1 \cdot n + mn - n + 2 = mn + 3$. The maximum value occurs when $i = n-1$ and $j = m$, giving $(n-1) + m \cdot n + mn - n + 2 = 2mn - n + m + 2$. When i is even, the minimum value occurs when $i = 2$ and $j = 1$, giving $mn + 1 \cdot n + 2 - n + 2 = mn + n + 3$. The maximum value occurs when $i = n-2$ and $j = m$, giving $mn + m \cdot n + (n-2) - n + 2 = 2mn + m$. Thus, the range of values for W_3 is the union of the intervals $[mn+3, 2mn - n + m + 2] \quad$ for odd i, $[mn + n + 3, 2mn + m] \quad$ for even i.

Now, we verify that $W_3 \subseteq W_1 \cup W_2$. We examine the ranges of W_1 and W_2. The range of W_1 for odd i is $\left[\frac{i+1}{2} + \frac{jn}{2} - \frac{n}{2} + mn + 1, 2mn + \frac{n(j-1)}{2} + \frac{i}{2} + 2\right]$, and for even i, it is $\left[2mn + \frac{n(j-1)}{2} + \frac{i}{2} + 2, \right]$. It can be observed that the range of W_3, $[mn+3, 2mn - n + m + 2]$ for odd i and $[mn + n + 3, 2mn + m]$ for even i, lies within the range of W_1.

The range of W_2 for odd i is $\left[\frac{mn}{2} + \frac{i+1}{2} + \frac{jn}{2} - \frac{n}{2} + mn + 1, \frac{5mn}{2} + 2 + \frac{i}{2} + \frac{n(j-1)}{2}\right]$, and for even i, it is $\left[\frac{mn}{2} + \frac{i+1}{2} + \frac{jn}{2} - \frac{n}{2} + mn + 1, \frac{5mn}{2} + 2 + \frac{i}{2} + \frac{n(j-1)}{2}\right]$. The range of W_3 is again contained within the range of W_2, as the lower and upper bounds of W_3 fit within the bounds of W_2. Therefore, since the range of W_3 is contained within the ranges of W_1 and W_2, we conclude that

$$W_3 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_4 \subseteq W_1 \cup W_2$. First, We compute the minimum and maximum values of the weight function W_4. The minimum value of $w_4(j)$ occurs when $j = 1$, giving $mn + 1 \cdot n - \frac{n}{2} + 2 = mn + n - \frac{n}{2} + 2 = mn + \frac{n}{2} + 2$. The maximum value of $w_4(j)$ occurs when $j = m$, giving $mn + m \cdot n - \frac{n}{2} + 2 = mn + mn - \frac{n}{2} + 2 = 2mn - \frac{n}{2} + 2$. Thus, the range of W_4 is the interval $\left[mn + \frac{n}{2} + 2, 2mn - \frac{n}{2} + 2\right]$.

Next, we verify that $W_4 \subseteq W_1 \cup W_2$. We examine the ranges of W_1 and W_2. The range of W_1 for odd i is giving $\left[\frac{i+1}{2} + \frac{jn}{2} - \frac{n}{2} + mn + 1, 2mn + \frac{n(j-1)}{2} + \frac{i}{2} + 2\right]$. The range of W_1 for even i is $\left[2mn + \frac{n(j-1)}{2} + \frac{i}{2} + 2, \right]$. By analyzing the range of W_1, we see that it contains the range of W_4, specifically, the lower bound of W_4 (which is $mn + \frac{n}{2} + 2$) fits within the lower bound of W_1, and the upper bound of W_4 (which is $2mn - \frac{n}{2} + 2$) fits within the upper bound of W_1.

The range of W_2 for odd i is $\left[\frac{mn}{2} + \frac{i+1}{2} + \frac{jn}{2} - \frac{n}{2} + mn + 1, \frac{5mn}{2} + 2 + \frac{i}{2} + \frac{n(j-1)}{2}\right]$, and for even i, it is $\left[\frac{mn}{2} + \frac{i+1}{2} + \frac{jn}{2} - \frac{n}{2} + mn + 1, \frac{5mn}{2} + 2 + \frac{i}{2} + \frac{n(j-1)}{2}\right]$. Similarly, the range of W_2 also contains the range of W_4, as both the lower and upper bounds of W_4 fit within the bounds of W_2. Since the range of W_4 is contained within the ranges of both W_1 and W_2, we conclude that

$$W_4 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_5 \subseteq W_1 \cup W_2$. First, we compute the minimum and maximum values of the weight function W_5. The minimum value of $w_5(i,j)$ occurs when $i = 1$ and $j = 1$, giving $2mn + 1 + 2 + n(1-1) = 2mn + 3$. The maximum value of $w_5(i,j)$ occurs when $i = n-1$ (the largest odd number less than or equal to $n-1$) and $j=m$, giving $2mn+(n-1)+2+n(m-1) = 2mn+n-1+2+n(m-1) = 2mn+n(m-1)+n+1$. Thus, the range of W_5 is the interval $[2mn+3, 2mn+n(m-1)+n+1]$.

Now, we verify that $W_5 \subseteq W_1 \cup W_2$. We examine the ranges of W_1 and W_2. For i odd, the range of W_1, giving $\left[\frac{i+1}{2}+\frac{jn}{2}-\frac{n}{2}+mn+1, 2mn+\frac{n(j-1)}{2}+\frac{i}{2}+2\right]$. For even i, the range of W_1, giving $\left[2mn+\frac{n(j-1)}{2}+\frac{i}{2}+2,\right]$. By inspecting these ranges, we can see that the values of W_5 lie within the range of W_1, as the lower bound of W_5 is greater than or equal to the lower bound of W_1 and the upper bound of W_5 is less than or equal to the upper bound of W_1. When W_2 for i odd, the range of W_2 is $\left[\frac{mn}{2}+\frac{i+1}{2}+\frac{jn}{2}-\frac{n}{2}+mn+1, \frac{5mn}{2}+2+\frac{i}{2}+\frac{n(j-1)}{2}\right]$.

For even i, the range of W_2 is $\left[\frac{mn}{2}+\frac{i+1}{2}+\frac{jn}{2}-\frac{n}{2}+mn+1, \frac{5mn}{2}+2+\frac{i}{2}+\frac{n(j-1)}{2}\right]$. Similarly, the range of W_5 is contained within the range of W_2, as both the lower and upper bounds of W_5 fit within the bounds of W_2. Since the range of W_5 is contained within the ranges of both W_1 and W_2, we conclude that

$$W_5 \subseteq W_1 \cup W_2.$$

We are tasked with proving that $W_6 \subseteq W_1 \cup W_2$. First, we compute the minimum and maximum values of the weight function W_6. The minimum value of $w_6(j)$ occurs when $j=1$, giving $2mn+2+\frac{n(2\cdot 1-1)}{2} = 2mn+2+\frac{n}{2} = 2mn+\frac{n}{2}+2$. The maximum value of $w_6(j)$ occurs when $j=m$, giving $2mn+2+\frac{n(2m-1)}{2} = 2mn+2+\frac{2mn-n}{2} = 2mn+2+mn-\frac{n}{2} = 3mn+2-\frac{n}{2}$. Thus, the range of W_6 is the interval $\left[2mn+\frac{n}{2}+2, 3mn+2-\frac{n}{2}\right]$.

Now, we verify that $W_6 \subseteq W_1 \cup W_2$. We examine the ranges of W_1 and W_2. For odd i, the range of W_1 is $\left[\frac{i+1}{2}+\frac{jn}{2}-\frac{n}{2}+mn+1, 2mn+\frac{n(j-1)}{2}+\frac{i}{2}+2\right]$. For even i, the range of W_1 is $\left[2mn+\frac{n(j-1)}{2}+\frac{i}{2}+2,\right]$. The range of W_1 for both odd and even i contains the range of W_6, as the lower bound of W_6 (which is $2mn+\frac{n}{2}+2$) fits within the lower bound of W_1, and the upper bound of W_6 (which is $3mn+2-\frac{n}{2}$) fits within the upper bound of W_1. For odd i, the range of W_2 is $\left[\frac{mn}{2}+\frac{i+1}{2}+\frac{jn}{2}-\frac{n}{2}+mn+1, \frac{5mn}{2}+2+\frac{i}{2}+\frac{n(j-1)}{2}\right]$. For even i, the range of W_2 is $\left[\frac{mn}{2}+\frac{i+1}{2}+\frac{jn}{2}-\frac{n}{2}+mn+1, \frac{5mn}{2}+2+\frac{i}{2}+\frac{n(j-1)}{2}\right]$. Similarly, the range of W_6 is contained within the range of W_2, as both the lower and upper bounds of W_6 fit within the bounds of W_2. Since the range of W_6 is contained within the ranges of both W_1 and W_2, we conclude that

$$W_6 \subseteq W_1 \cup W_2.$$

Therefore, the number of distinct edge weights is $|W_1 \cup W_2| = |W_1| + |W_2| = mn + mn = 2mn$.This implies that $rac(Amal(DW_n, v, m)) \leq 2mn$. Since $rac(Amal(DW_n, v, m)) \geq 2mn$ and $rac(Amal(DW_n, v, m)) \leq 2mn$, we conclude that $rac(Amal(DW_n, v, m)) = 2mn$.

For any two vertices, there exists a rainbow path, as presented in Table 3.

TABLE 3. $y-z$ Rainbow path in amalgamation of double wheel graph $Amal(DW_n, v, m)$

y	**z**	**rainbow path**
$x_{i,j}$	v	$x_{i,j}, v$
$y_{i,j}$	v	$y_{i,j}, v$
$x_{i,j}$	$y_{i,j}$	$x_{i,j}, v, y_{i,j}$
$x_{i,j}$	$x_{k,l}$	$x_{i,j}, v, x_{k,l}$
$y_{i,j}$	$y_{k,}$	$y_{i,j}, v, y_{k,l}$
$x_{i,j}$	$y_{k,l}$	$x_{i,j}, v, y_{k,l}$

It can be easily observed from the table above that for any two vertices, there exists a rainbow path between them, which satisfies the condition for a valid rainbow antimagic coloring. □

As an example can be viewed in Figure 3.

4 Practical Applications The exact values of the rainbow antimagic connection number $rac(G)$ obtained in this paper provide practical design guidance for systems that require non-repeating and verifiable path signatures on modular network topologies. In our setting, a bijective vertex labeling induces edge-weights through $w(uv) = f(u) + f(v)$, and rainbow connectivity guarantees the existence of

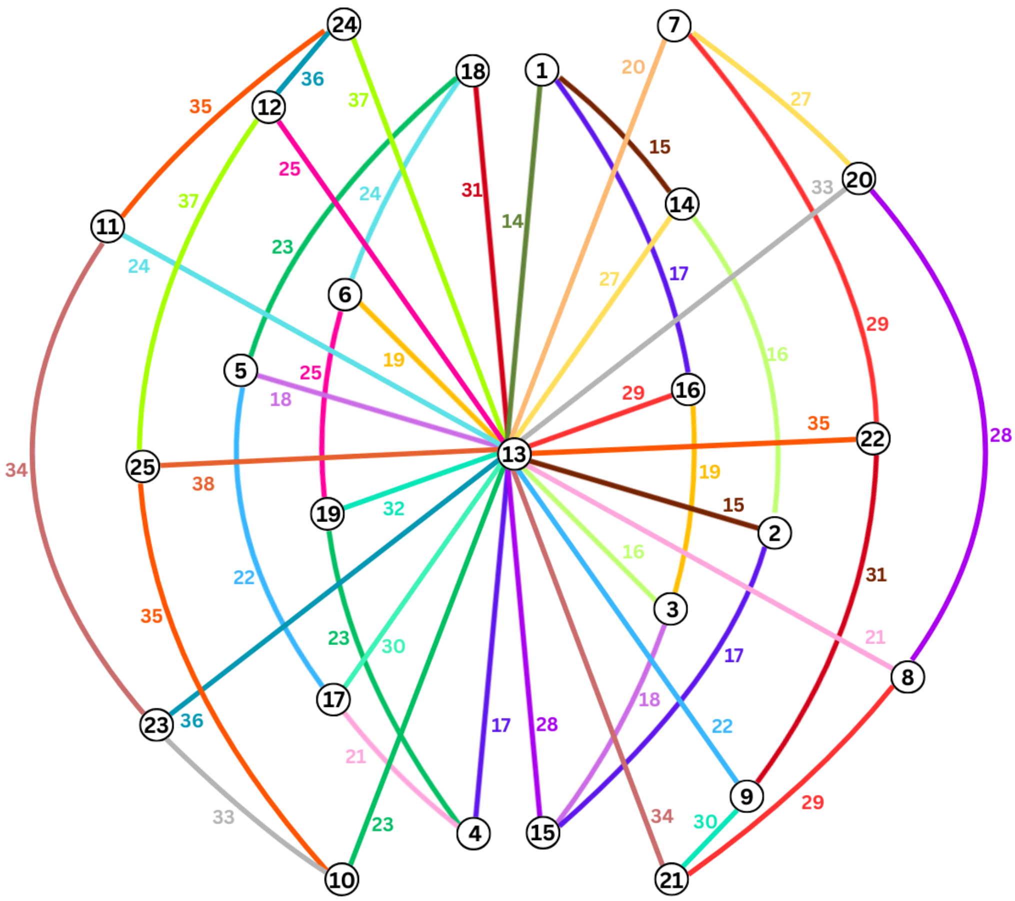

FIGURE 3. A rainbow antimagic coloring of $rac(Amal(DW_6, v, 2))$

a path whose edge-weights are pairwise distinct between any two vertices. From an application-oriented perspective, such rainbow paths can be interpreted as deterministic route signatures, because the ordered sequence of induced edge-weights along a selected connection can act as an identifier that is reproducible from the underlying labeling rule yet difficult to replicate without access to the same construction.

This interpretation becomes particularly relevant for hub-based modular infrastructures, where a large system is composed of multiple similar submodules attached through a common linking vertex. Amalgamation models precisely this situation, since several copies of a base graph share a central vertex, yielding a structure that naturally resembles modular communication clusters, hub-and-spoke distribution systems with repeated local cycles, or merged wheel-like subnetworks. In such settings, it is often desirable to generate distinguishable connection patterns that support monitoring, traceability, and the localization of faults or anomalies. The closed-form formulas for amalgamations of sunflower, lemon, and double wheel graphs directly quantify how the minimum diversity of induced edge-weights must scale in order to preserve rainbow connectivity as the number of modules m and their size n increase. Hence, the obtained expressions can be viewed as explicit scalability rules. They translate structural growth parameters (m, n) into the minimal signature diversity required to maintain non-repeating weight patterns along connecting paths.

Beyond the network interpretation, the induced edge-weights produced by RAC can also be leveraged for document integrity and authentication. In particular, RAC-based induced edge-weight sequences can be used to construct a structured watermark for tamper-evident verification. The key idea is to use the RAC construction to generate a deterministic sequence $s = [w(e_1), w(e_2), \ldots]$ derived from selected edges, where the distinctness properties associated with rainbow paths support the formation of non-repeating patterns. The resulting sequence can then be encoded into a fixed-length binary stream and embedded into Police Clearance Certificate images using a Least Significant Bit (LSB) embedding scheme. At the verification stage, authentication is performed by extracting the embedded bits, reconstructing the recovered sequence, and checking whether it matches the original RAC-derived watermark pattern. This workflow provides a transparent integrity check mechanism, since any significant alteration of the host image that disrupts the embedded bitstream is likely to produce a mismatch in the reconstructed sequence.

The exact $rac(G)$ values obtained in this paper provide a principled way to parameterize both the watermark length and the diversity of available watermark patterns for a chosen underlying topology. A larger $rac(G)$ corresponds to a larger space of distinct rainbow weight patterns that can be exploited as watermark candidates, which is relevant when a system requires multiple independent watermark instances while maintaining a uniform construction rule. Moreover, the amalgamation results clarify how this diversity scales predictably with m and n, enabling an informed selection of parameters that balances complexity, watermark length, and verifiability. For example, increasing the number of amalgamated modules m or the module size n enlarges the induced weight space in a controlled manner, which can increase the pool of feasible watermark patterns under the same RAC framework.

More explicitly, the closed-form expressions $rac(\text{Amal}(Sf_n, v, m)) = 3mn$ and $rac(\text{Amal}(Le_n, v, m)) = rac(\text{Amal}(DW_n, v, m)) = 2mn$ provide an explicit capacity-scaling description for RAC-derived pattern generation on these amalgamated families. Under the induced-weight framework, $rac(G)$ captures the minimum diversity level needed to ensure rainbow connectivity throughout the topology. Accordingly, it can be interpreted as a quantitative indicator of how the space of admissible non-repeating rainbow weight patterns grows as the structure expands. In particular, enlarging the underlying graph by increasing the number of amalgamated modules m or the module size n leads to a linear scaling of this indicator, with coefficient 3 for the sunflower amalgamation and coefficient 2 for the lemon and double wheel amalgamations.

From a parameter-selection viewpoint, these formulas help relate structural choices (m, n) to the expected scaling of the available RAC-derived patterns that may be used as route signatures or watermark candidates under a consistent construction rule. For fixed n, increasing m increases the indicator proportionally, reflecting predictable growth of the feasible pattern space across a larger number of modules; similarly, for fixed m, increasing n yields proportional scaling as each module becomes structurally richer. The difference between the coefficients $3mn$ and $2mn$ highlights that these families exhibit distinct linear scaling rates with respect to the same parameters, which may be useful when selecting a graph family to match application-specific constraints on pattern diversity, construction complexity, and verification needs.

From a security viewpoint, the RAC-derived watermark construction is attractive because its verification relies on matching a structured pattern rather than on a single fragile feature. Conceptually, this supports tamper evidence in common scenarios such as partial cropping, local editing, or re-compression, where unintended modifications may perturb the embedded bits and cause a detectable mismatch. While the robustness characteristics of a specific LSB implementation depend on image handling and post-processing operations, the graph-theoretic layer provided by RAC offers an additional source of structured complexity that is governed by explicit formulas. Therefore, our theoretical results can be used as design guidelines to select graph parameters that deliver an adequate diversity of watermark patterns while keeping the verification procedure simple and reproducible.

We emphasize that the present discussion is intended to highlight practical relevance rather than to introduce new performance claims. Further work may investigate implementation level robustness under concrete attack models, compare RAC-based watermarks with alternative constructions, and explore additional graph families beyond amalgamated wheel structures for application-specific constraints.

5 Concluding Remarks This research presents the concept of rainbow antimagic coloring, a fusion of antimagic labeling and rainbow connection in graph theory. We have explored the properties of rainbow antimagic coloring and provided new results regarding its application to various graph families. Specifically, we determined the rainbow antimagic connection number, $rac(G)$, for different graph amalgamations, including sunflower, double wheel, and lemon graphs. We have demonstrated that the rainbow antimagic coloring of amalgamated graphs can be computed by leveraging bijective functions to assign edge weights and ensuring the uniqueness of vertex sums. The results showed that for specific graph families, such as $Amal(Sf_n, v, m)$, $Amal(Le_n, v, m)$ and $Amal(DW_n, v, m)$, the rainbow antimagic connection number is equal to $3mn$, $2mn$ and $2mn$, respectively, providing exact values for these previously unexplored cases. Moreover, we proved that for several graph amalgamations, such as the sunflower and double wheel graphs, the edge weights associated with rainbow antimagic coloring are distinct and fulfill the conditions for a valid rainbow path. This is crucial for ensuring the correctness of rainbow antimagic coloring, which is characterized by the uniqueness of edge weights across paths connecting any two vertices. The findings of this study contribute to the expanding body of knowledge in graph coloring and labeling theory, providing a robust framework for future investigations into rainbow antimagic coloring in more complex graph structures In addition to the theoretical contributions, we briefly outlined a practical implication of RAC in document authentication. Specifically, the induced edge-weight sequence can be used to construct a verifiable watermark for Police Clearance Certificate images via an LSB embedding and extraction matching procedure. The closed-form dependence on m and n further provides a scalable guideline for selecting parameters that balance watermark diversity and verification requirements.

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