Journal of Information Hiding and Multimedia Signal Processing ©2019 ISSN 2073-4212
Ubiquitous International Volume **10**, Number **2**, March **2019**

A Survey on 3D Face Recognition based on Geodesics

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Received October 2017; revised August 2018

Abstract. *With the development of computer vision technology, the research on three-dimensional face recognition has been carried out in many aspects and there have been a large number of works on face recognition algorithms and applications. As an important intrinsic geometry structure on the surface, geodesic is becoming an influential mathematical tool of three-dimensional face recognition. After introducing the concept and algorithms of geodesic and 3D face data acquisition and database, this paper elaborates the basic idea and summarizes algorithms of 3D face recognition based on geodesics both in small expression variation with isometric deformation and in large expression variation with non-isometric deformation. At last, the paper points out the future research direction of 3D face recognition. The three-dimensional face recognition method based on geodesics is important to judge the similarity of the face, the public security criminal investigation, the confirmation of kinship and anthropological research. Thus, it has prominent theoretical value and practical significance.*
Keywords: 3D face recognition, Geodesic, 3D face database

1. **Introduction.** Face recognition is a kind of biometrics recognition technology, which uses human face feature information on identification. With the development of computer vision technology, the research on three-dimensional(3D) face recognition has been carried out in many aspects and there have been a large number of face recognition algorithms and applications [1, 2], especially the two-dimensional (2D) face recognition rate is high in many cases. However, there are many difficulties in 2D face recognition on dealing with changes in light, head posture, facial expression, occlusion, and etc. 3D face recognition has advantages on these problems. At the same time, with the popularity of three-dimensional scanning devices, it is becoming more and more convenient to acquire high-precision 3D face data. The appearance of large amounts of 3D face data has provided

the data foundation for the related research on 3D face recognition, which has gradually become a hot spot.

Three-dimensional face recognition can overcome the influence of light changes and eliminate the influence of pose variations after data preprocessing and registration. Therefore, overcoming the impact of facial expression variations, age and occlusion is existing main issue in the 3D face recognition. Considering that the geodesic distance is preserved under isometric deformation, and the geodesic is intrinsic, which can overcome the influence of facial expression changes and has been widely used in 3D face recognition, this paper will summarize the related algorithms of 3D face recognition based on geodesics and point out the future research direction.

2. Geodesic.

2.1. The basic concept of geodesic.

The geodesic stems from geodesy and was originally used to measure the earth. Geodesic can be seen as a spread of a straight line in the bending space, which is used to measure the shortest path length between the two points of the surface. It is an important concept of the differential geometry and Riemannian geometry. As an influential intrinsic geometry tool, geodesic has been widely used in computer graphics, image processing, computational geometry, computer vision (Such as shape analysis, shape matching, parameterization and texture mapping, and etc.) 3D mesh model segmentation, face recognition, fluid movement on surface and many other applications have achieved better results due to the introduction of geodesic.

(1) Geodesic on a smooth surface

In the differential geometry, the geodesic is defined as a curve with a geodetic curvature of zero [3]. In the Riemannian manifold space, the geodesic is defined as:

Definition 2.1. *A smooth curve* $\gamma : I \to M$ *in an affine connection space* (M, D) *is called a geodesic if its tangent* $\gamma'(t)$ *is a parallel tangential field along* γ*, that is* $\frac{D}{dt}(\frac{d\gamma}{dt}) \equiv D_{\gamma'(t)}\gamma'(t) = 0$. [4]

Geodesic in the Euclidean space is the shortest. In the general Riemannian manifold, the geodesic is only the local minimum. As the shortest path, geodesic is defined as follows in Riemannian geometry:

Definition 2.2. *Assuming a smooth curve* $\Gamma : [\mathrm{a}, \mathrm{b}] \to \mathrm{M}$ *on the Riemannian manifold M, the arc length of* Γ *is given by the equation (1).*

$$\mathrm{L}(\Gamma) = \int_a^b \left\|\dot{\Gamma}(t)\right\|_2 dt = \int_a^b \sqrt{\left\langle \dot{\Gamma}(t), \dot{\Gamma}(t) \right\rangle_{\Gamma(t)}} \mathrm{dt} = \int_a^b \sqrt{g_{\Gamma(t)} \left\langle \dot{\Gamma}(t), \dot{\Gamma}(t) \right\rangle} \mathrm{dt} \quad (1)$$

The extreme value of the function $\mathrm{L}(\Gamma)$ *(i.e., the minimum or maximum) is called the geodesic, and the minimized path is called the* ***shortest geodesic*** [5].

From the famous HopfRinow theorem, we can see that there exists a shortest path between each pair of points in the connected and compact Riemannian manifold space, which is complete. So we can define the measure of the length, which is called a geodesic measure.

$$\mathrm{d}_X(x_1, x_2) = \min_{\Gamma}\{L(\Gamma)\mathrm{s}.t.\Gamma : [a, b] \to X, \Gamma(a) = x_1, \Gamma(b) = x_2\} \quad (2)$$

In this definition, it is derived from the local Riemann structure. It is an intrinsic definition, because it is not limited to the specific embedded space. As can be seen from above, the geodesic has the following three mutually equivalent definitions: 1)Geodesic is a locally shortest curve. 2)Geodesic has vanishing geodesic curvature. 3)Geodesic is parallel to the surface tangent vector when it moves along the affine connect.

(2) Geodesic and algorithms on polyhedral surfaces

On polyhedral surfaces, the curve satisfied that it is the local shortest curve is called the shortest discrete geodesic, namely, the shortest geodesic. The curve satisfying that the geodetic curvature is zero is called the straightest geodesic. Because the shortest path satisfies the properties of the metric, it is widely used in many fields. So we will focus on the shortest geodesic algorithms.

Computing an exact geodesic on the smooth surface by solving the geodesic differential equation has high computational complexity and can only be applied to the parametric surface. So the later studies try to apply the special properties of discrete geodesics instead of the geodesic differential equation in the pure mathematical methods to solve geodesic. Discrete geodesic problems have appeared in the form of calculating the geometric shortest path problem in the early years. Sharir and Schorr [6] firstly proposed the shortest path problem on convex polyhedron surfaces in 1986, and Mitchell et al.[7] first studied the three-dimensional shortest path problem on general polyhedral surfaces, which is called Discrete Geodesic Problem (DGP). Geodesic algorithm can be divided into two types according to the method and result of the solution: the exact shortest geodesic algorithms based on computational geometry, and the approximate shortest geodesic algorithms based on PDE.

The exact shortest geodesic algorithms based on computational geometry compute the shortest path between the two points on the polyhedral mesh surface by the geometric analysis of the polyhedral mesh. In TABLE 1, there are four exact geodesic algorithms and their time complexities are listed. MMP algorithm of Mitchell et al. [7] is the most classical geodesic algorithm on a triangular mesh surface. And in the latest years, improved CH (ICH) algorithm [10] and parallel CH (PCH) algorithm [11] are proposed and greatly improves the computational speed. Ying et al. [12] also proposed a new solution for discrete geodesic problemsSaddle Vertex Graph (SVG) in 2013 and can effectively reduce time complexity of the algorithm.

Table 1. The comparison of classic geodesic algorithms

Object	Type	Algorithm	Time complexity
General polyhedron surface	Exact algorithms	O'Rourke [8]	O(n^5)
		MMP[7]	O($n^2 logn$)
		CH[9]	O(n^2)
		SVG[12]	O(Dnlogn)
	Approximate algorithms	Dijkstra's[13]	O($\|E\|log\|E\|$)
		FMM[14]	O(nlogn)
		Heat method[16]	

Geodesic algorithms based on PDE method compute the discrete geodesic by solving the partial differential equation to obtain the approximate solution. This kind of method is relatively easy and fast to realize, but can only get approximate geodesics. It can be applied in the fields with low accuracy and high speed. Dijkstra algorithm[13] is used to find geodesic on the graph in the early, but its accuracy is too low. In 1996, Sethian[14] proposed an first-order approximation Fast Marching Method (FMM) for the general grid to computed geodesics with the time complexity of O(nlogn). In 1998, Kimmel and Sethian[15] extended the Fast Marching Method to triangular meshes. In 2013, Crane et al.[16] proposed a new method for calculating the geodesic distance based on heat flow. The algorithm is robust, efficient, simple and convergent to the exact distance. After a preprocessing, the result can be calculated in the near linear time. Xin et al. [17] proposed the method of reconstructing the geodesic from the gradient, which is not only applicable to the complete surface, but also the defective surface.

3. **3D face data acquisition and database.**

3.1. **3D face data acquisition.**

With the development of computer vision technology and hardware equipment, 3D face data acquisition is more and more convenient. At present, the methods for obtaining 3D face data mainly include three-dimensional laser scanning, structured light scanning, CT, ultrasonic measure and etc. The 3D laser scanner obtains the 3D data and texture information of human face through the triangulation principle, which is the most direct method of collecting the face geometry data. The laser scanner can capture high-precision face geometry data, but scanning effect is poor on the part of lower reflectivity (such as hair), and for areas with obstructions, such as eyes and teeth, effective data cannot be obtained. The structured light scanner mainly emits laser beams onto objects, and the camera captures the bright bars to collect 3D face information. The scanning speed of the structured light is fast;the acquired information is complete and accurate with little affected by the light, but the device is bulky, and scanning scope is limited. CT scanners mainly use CT medical imaging devices to scan the faces, and the head of the slice data can be obtained after scanning. In summary, three-dimensional photographic scanning can only obtain epidermis data. The slice data collected by CT scan includes both surface information and internal structure information. Ultrasonic measurement can only obtain feature point data and requires secondary synthesis. With the development of 3D acquisition equipment and the increasing demand for 3D face data, more and more research organizations began to build their own 3D face database. The 3D face database is shown in TABLE 2.

4. **3D face recognition based on geodesics.**

Three-dimensional face recognition methods can be classified into two types: global feature-based method and local feature-based method. The 3D face recognition methods based on the global feature mainly recognize a human face from the global feature. The global methods can be classified into three kinds of method. The first one is spatial information method, in which the 3D face surface similarity is compared directly between two aligned face surfaces (such as the Iterative Closest Point,ICP [24, 25, 26] and Hausdorff distance methods [27, 28], and etc.). The second one is the surface distance method, in which surface geodesic induced by the Riemannian metric is used to construct an expression-independent face representation. And the third one is deformation model method [29], which simulates the deformation between faces through some physical principles(such as thin plate spline (TPS) fitting [30], the fitting of the Annotated Deformable Model [31] and the fitting of the Deformable model [32]). 3D face recognition based on local feature method is generally used to deal with local shape deformation due to facial expression changes. This method [33, 34, 35, 36, 37, 38] divides the face into many areas by facial markers and then identifies the individual regions, and finally fuses all the results.

As an important intrinsic geometry structure on the surface, it is becoming an influentia mathematical tool of three-dimensional face recognition. Therefore, the 3D face recognition method based on the geodesics (surface distances) has become an prominent method of face recognition. At present, most of the 3D face recognition methods based on geodesic (surface distance) assume that the 3D face model with different facial expressions can be regarded as isometric, which will be discussed in detail as follows.

Table 2. 3D face database

Name	Data Type	Data Size	Quantity	3D Models / Person	Variety	Texture	Website
Gavadb [18]	Triangular mesh	10000 ∼ 20000 points	61 person Male 45 Female 16	9	Pose Expression	No	http://www.gavab.es
BU-3DFE [19]	Triangular mesh	3000 ∼ 10000 points	100 person Male 44 Female 56	25	Expression	Yes	http://www.cs.binghamton.edu/∼ lijun/Research/3DFE
Gavadb [18]	Triangular mesh	10000 ∼ 20000 points	61 person Male 45 Female 16	9	Pose Expression	No	http://www.gavab.es
CASIA	Triangular mesh	10000 ∼ 20000 points	123 person	37∼38	Pose Expression Illumination	Yes	http://www.cbsr.ia.ac.cn/china/Gait%20Databases%20CH.asp
BJUT [20]	Triangular mesh	13×10^4 points	500 person	1	Neutral expression	Yes	http://www.bjut.edu.cn/sci/multimedia/mul-lab/3dface/face_database.htm.
3DFED	Triangular mesh	25000 points	40 person	9	Expression		
Bosphorus [21]			105 person Male 60 Female 45	31 ∼ 254	Pose Expression Occlusion	Yes	http://bosphorus.ee.boun.edu.tr/default.aspx
FRGC V2.0		320×240	465 person		Expression		http://www.frvt.org/FRGC/
Texas 3D [22]	Range image	501×751	118 person		Pose Expression Illumination	Yes	http://live.ece.utexas.edu/research/texas3dfr/
UND [23]	Range image	320×240	277 person		Front Expressionless		http://www3.nd.edu/∼cvrl/CVRL/Data_Sets.html
USF			100 person		Neutral expression		http://www.csee.usf.edu/∼sarkar/index_files/DataAndCode.htm
3D RMA	Point cloud data	4000 points	120 person Male 106 Female 14	3	Pose	No	http://www.sic.rma.ac.be/∼beumier/DB/3d_rma.html

4.1. **3D face recognition algorithms based on geodesics in isometric deformation.** The basic idea of 3D face recognition based on geodesics in isometric deformation is to assume that a face surface can be expressed as a compact, connected, zero-genius 2-Riemannian manifold embedded in R^3, denoted by S. S_1 and S_2 are 3D face surfaces of the same person with two different expressions, which S_1 is a face surface of neutral expression, S_2 is the face of other facial expressions. $f : S_1 \to S_2$ is the differential homeomorphism of the modeling expression. Based on this hypothesis, the surface geodesic induced by the Riemann metric, as an intrinsic geometry, is used to construct an expression-independent face representation.

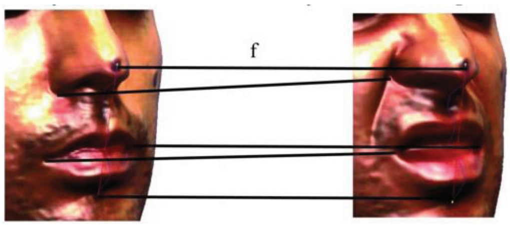

FIGURE 1. The correspondence between faces of different expressions [39]

3D face recognition methods based on geodesics in isometric deformation assume that the 3D face model with different facial expressions can be regarded as isometric. This hypothesis transforms the face contrast problem under different expressions into an isometric surface matching problem. Bronstein et al.[40] placed 133 mark points on the human face to calculate the distance between these points under different expressions. The results show that the geodesic distance is more stable than Euclidean distance under different expressions. In other words, this assumption means that f keeps the geodesic distance between each pair of points on the face surface constant, i.e., $d_{S_1}(x, y) = d_{S_2}(f(x), f(y)), \forall x, y \in S_1$. The hypothesis is effective in small facial expressions, that is, under small expression changes, different expressions of the same person's 3D face surfaces can be seen as isometric deformation in which the geodesic distance remain unchanged.

The 3D face recognition methods based on geodesics (surface distances) in isometric deformation have been explored by several research groups (such as Bronstein et al., Mpiperis et al. and Berretti et al.). According to the different surface parts used for face matching, this method can be divided into three categories: (a) Iso-metric surface method, which matches the face with the entire iso-metric surface. (b) Iso-geodesic stripes method, that is, face is matched on a set of iso-geodesic stripes with equal width. (c) The method based on geodesics, the geodesic curves are used as the facial feature for the face matching.

1)Iso-metric surface method

Bronstein et al. [41] assumed that the different facial expressions models of the 3D faces could be regarded as isometric deformations, under which the geodesic distances are constant, so they use various methods based on geodesic distances to analyze and recognize 3D human face. In reference [42], Bronstein et al. used isometric tensor surfaces to calculate the geodesic distance. In reference [43], Bronstein et al. used a combined 3D classical form for multimodal 2D + 3D face recognition. In [44], Bronstein et al. discussed the best embedding space and demonstrated examples of twins recognition. Bronstein et al. studied the embedding geometry and dimension selection in [45]. In reference [46], Bronstein et al. measured the similarity between faces iso-metrically by embedding a face surface into another one. Ahdid et al. [47] proposed a three-dimensional face recognition system based on the geodesic distance matrix from the reference point to the other points on the 3D face surface. Quan et al. [48] proposed a face recognition method based on a mapping formed by the geodesic distances between the twelve critical points to all surface points on the face. The presented representation can capture all the geometric information from the entire 3D face, and provide a compact, expression-independent mapping that keeps the intrinsic geometry information. Ouji et al. [49] proposed a 3D face recognition using R-ICP and geodetic coupled approach (FIGURE 2). First, the R-ICP (Region-based Iterative Closest Point) method is used to align the static area of the face. And then two geodesic maps are calculated for the pair of vertices in the matched face regions. Finally, the distance between these maps became the foundation of recognition and authentication similarity scores. Their assessment experiments were completed on the 3D face data set of the IV2 French project with a global recognition rate (RR) of 87.2%. Vezzetti et al. [50] proposed an automatic 3D face recognition strategy based on geometric descriptors and landmarks, which is based on the differential geometry theory. First, 17 feature points are automatically extracted according to the geometrical properties of the facial shape. Then, a set of geodesic and Euclidean distances (FIGURE 3), together with nose volume and ratios between the geodesic and the Euclidean distances has been computed and accumulated in a final score to compare faces. These methods can not handle the large expression variation which is a non-isometric deformation. In this case, the assumptions are not true that the geodesic distance is irrelevant isometric.

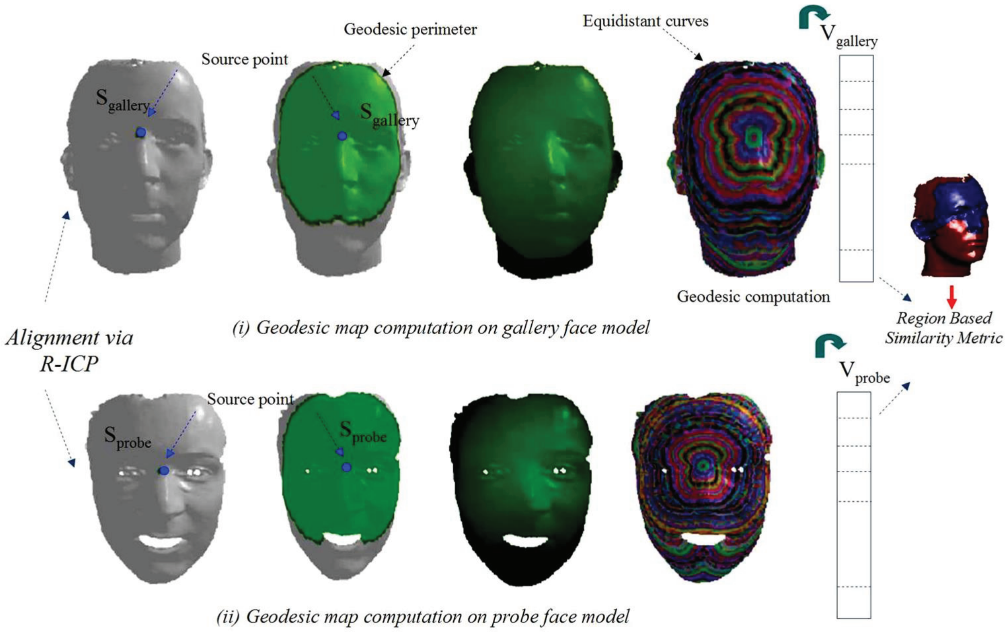

FIGURE 2. Geodesic maps computation [49]

2)Iso-geodesic stripes method

Berretti's research team has proposed a kind of method based on iso-geodesic stripes [51, 52, 53, 54] for three-dimensional face recognition. This kind of method represents a facial surface as a graph or sub-graph of iso-geodesic stripes with equal width and numerals and calculates the matching of the two graphs to compare the two faces. As shown in FIGURE 4, the main steps of their method are as follows: Firstly, the facial surface is divided into iso-geodesic stripes and sub-stripes with equal width and numerals. Secondly, the distances between each pair of iso-geodesic stripes (or sub-stripes) are calculated through the 3D Weighted Walkthroughs (3DWW). Thirdly, a facial surface is represented

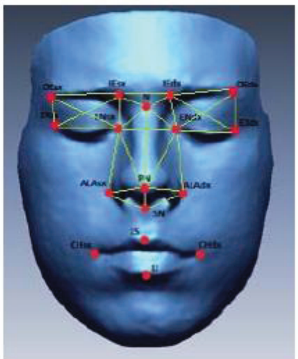

Figure 3. The detected subset of 28 geodesic distances [50]

as a graph, where the nodes represent stripes (or sub-stripes), edges represent distances between the stripes (or sub-stripes). Fourthly, the similarity of two facial surfaces is computed through the match of the two graphs.

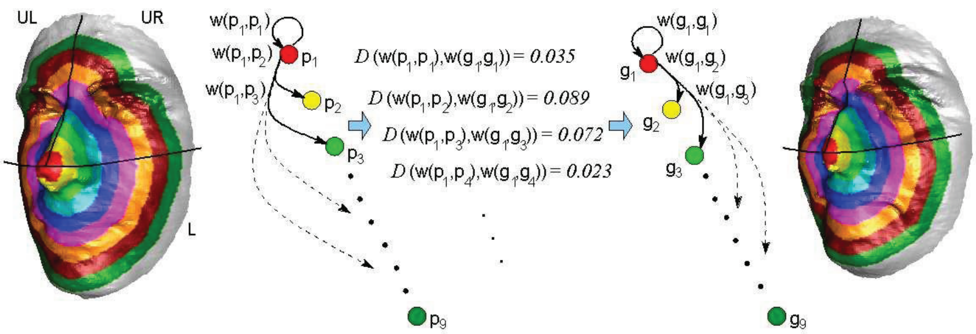

Figure 4. Iso-geodesic stripes method [53]

In 2011, Li et al. [55] made a similarity measure of the three-dimensional face by extracting iso-geodesic stripes on the 3D face model and comparing the spatial distribution vector in the iso-geodesic stripes. Based on the geodesic distance from the nose point, a group of iso-geodesic stripes are extracted from the 3D face model, and then the correlation coefficients of the spatial distribution relation matrices on the two 3D face models are calculated for measuring the similarity of them. This method improves the spatial distribution relation between two surface entities from the differential calculation in [56] to discrete statistics in order to improve the computational efficiency of the algorithm.

This kind of method is robust to expression variations for two main reasons: (a) The characters of iso-geodesic stripes are not much changed under the facial expression variations. (b) The 3DWW does not change much when the points on the iso-geodesic stripes are slightly moving or deforming because of the continuous character of the 3DWWs. The recognition rate of Rank-one is 99.53% on the SHREC08 face database and 94.1% on the FRGC v2.0 library. This kind of method is suitable for the little expression variations under isometric deformation. For the large expression variations under non-isometric deformation where the expression is stretched or compressed, the geodesic distances cannot be reserved, and the similarity cannot be measured exactly. This kind of method also cannot process the face models with opening mouth, where the topology of the model has been changed and the genus of the model is not zero.

3)The face recognition method based on geodesics or iso-geodesics

The face recognition methods based on geodesics or iso-geodesics are mainly based on a group of geodesics, iso-geodesics or their combinations extracted from the 3D face models for face recognition and similarity calculation. Ter Haar and Veltkamp [57, 58] performed face recognition based on facial contour curves extracted by geodesic distance and the similarity of the curves. Jahanbin et al. [59] established a framework for identity authentication based on the equal depth curves and iso-geodesics extracted from the face surface. In 2014, Jahanbin et al. [60] proposed a passive three-dimensional face recognition method using isometric contours and Prussian analysis, which represents 3D face by the iso-geodesics extracted from around the three source points: the nose tip(NT), the left-eye inner corner (LEIC) and right-eye inner corners (REIC) . And then the radial Euclidean distances, curvatures along the contour, distances from a Procrustean circle and distances from the Procrustean ellipses are calculated. Finally, a unified face recognition system is established by the stratified scheme combined with contour features. The hierarchical scheme is as follows: first, the individual evaluation of each feature which is extracted from the same marker points. Then, the fusion features extracted from the same mark points produce three parallel face recognition systems. Finally, the three face recognition systems are merged into a unified face recognition system. The recognition rate of the recognition system after fusion is up to 99.86%.

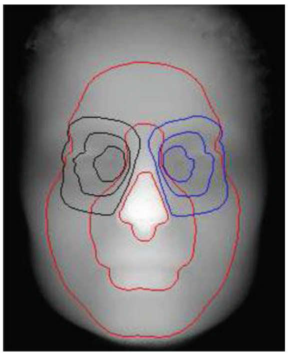

FIGURE 5. Example of a range image with iso-geodesic contours extracted from around NT, LEIC, and REIC [60]

Mpiperis et al.[61] proposed 3D face recognition methods based on geodesic polar representation constructed by geodesics and iso-geodesics, and their approach is superior to Bronstein et al. [45]. Compared with [45], they proposed a fast geodesic distance calculation method, and at the same time they mapped their lips to a continuous geodesic area to make the problem of the opening mouth improved. The parameterization of the polar geodesic coordinate in this method depends on solving a differential equation, and the precision of the solution depends upon the surface sampling or resolution. In 2014, Mpiperis et al.[62] also proposed an expression-Compensated 3D face recognition technique with geodesically aligned bilinear models. Firstly, the face surface is represented by a parametric surface model, which is described by a fixed-length parameter vector. A dense corresponding domain is defined between the model and the face by the geodetic polar parameter (FIGURE 9), and then the fitting is performed iteratively. Model parameters are estimated by minimizing the energy function and finally the face manifold is constructed by the bilinear model. The bilinear model allows the separation of

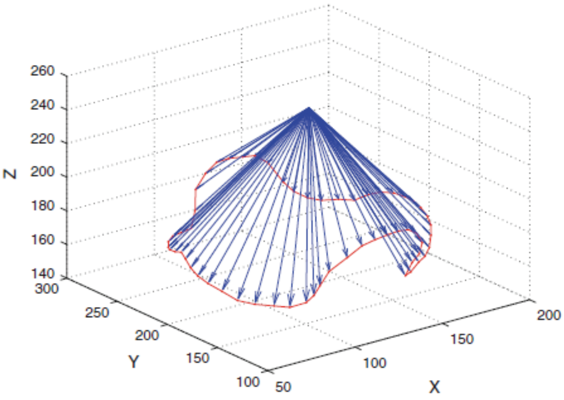

FIGURE 6. Radial Euclidean distances between the nose tip and equal arc length sample points on an iso-geodesic facial contour [60]

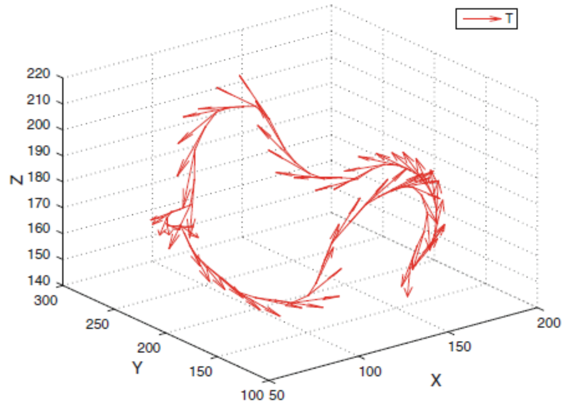

FIGURE 7. Example facial contour around the nose tip and tangent vectors along the path [60]

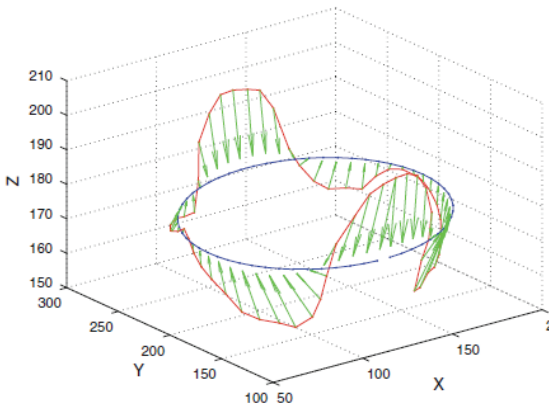

FIGURE 8. Example facial contour around the nose tip with arrows showing the distance between corresponding points on the contour and the Procrustean fitted circle [60]

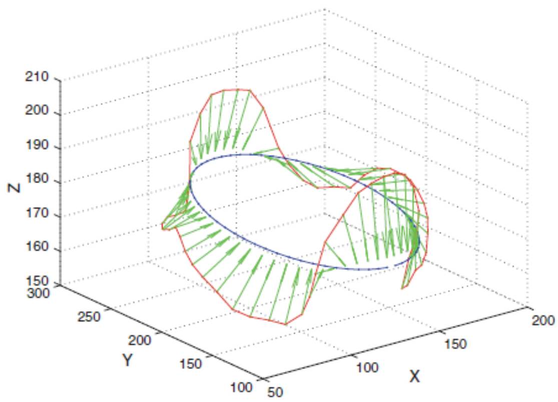

FIGURE 9. Example facial contour around the nose tip with arrows showing the distance between corresponding points on the contour and a Procrustean fitted ellipse [60]

the identity and facial appearance effects, and encodes their contributions with individually controlled parameters. The method uses the geometric coordinates to construct an expression-independent attribute image for face recognition on the BU-3DFE database. The increased recognition rate shows that the bilinear model can handle the expression well, which is the main flaw of current 3D face recognition systems.

Zhao et al. [63] proposed a similarity measure method of 3D face based on geodesic grids and curvature features using the exact geodesic distance algorithm on the surface,

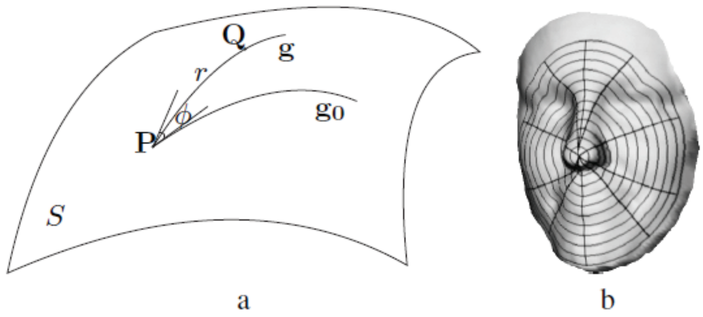

FIGURE 10. Geodesic polar parameterization of a surface.[62](a) Definition of geodesic paths and circles. (b) Geodesic paths and circles over a face surface

which improved the calculation accuracy of geodesic distance and geodesic. Based on the pre-processing of each 3D face model, the method firstly extracts the geodesics and iso-geodesics to establish the geodesic grid on the 3D face model (As shown in FIGURE 11). And then the absolute value of correlation coefficients of the four weighted average curvature features (mean curvature, Gaussian curvature, shape index and curvedness) of the 3D face model on neighborhood of the geodesic corresponding grid point are calculated to compare the two 3D face models.

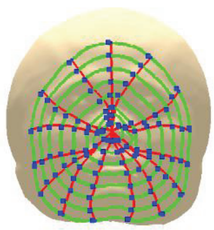

FIGURE 11. Geodesic grid points on the 3D face [63]

This kind of method has the following drawbacks: (a) The assumption of iso-metric surface deformation is ineffective in the case of extreme expression variations. (b) Face recognition results are affected by the position of the nasal tip which is detected automatically. (c) The problem of the opening mouth is also difficult to solve.

4.2. **3D face recognition algorithms based on geodesics in non-isometric deformation.** Large facial expressions can cause combination changes of stretch and contraction. In this situation, the geodesic distance may be changed. The following figure shows the change of the geodesic distances and the Euclidean distances between four mark points on a face tracking the expression changes.

In FIGURE 12, the Euclidean distance is changed from 113 mm to 103 mm, and the geodesic distance is changed from 115 mm to 106 mm. And they all have increased in the other two examples. This clearly shows that large facial expressions can cause stretch and contraction of facial shapes. In this elastic deformation, Euclidean distance and geodesic distance are not maintained. So the assumption that the face is isometric is not always right. And geodesic distances are more stable than Euclidean distances. For the

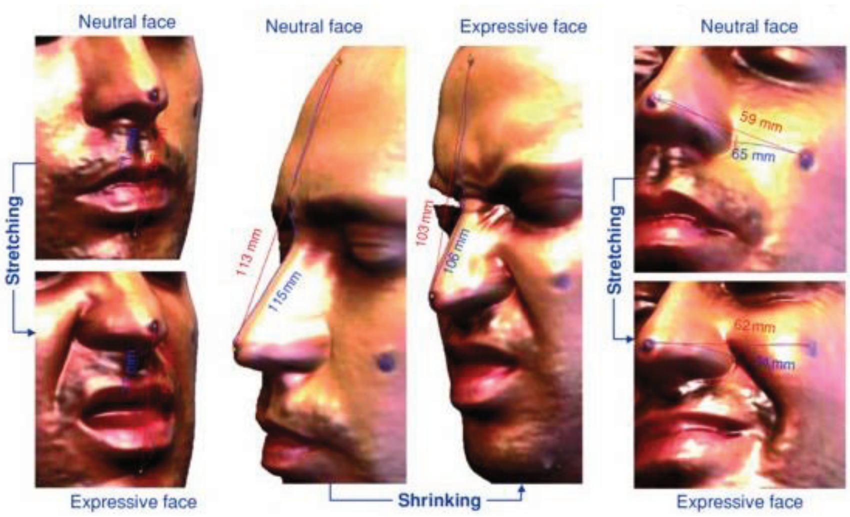

FIGURE 12. Variation of Euclidean distance (red line) and geodesic distance (blue line) in different expressions of 3D face models [64]

large expression changes of non-isometric deformation is one of the difficulties of 3D face recognition. There are some new approaches to address this situation face recognition and have good results. These new methods use elastic shape analysis to deal with facial elastic deformation caused by facial expression variations.

Based on the surface shape analysis theory and the Riemannian geometry theory, Daoudi et al. [65] used the face radial curve to represent the face, and then transformed these curves into the shape space to measure the similarity of the curves by the geodesic distance for face recognition. In the neutral expression of face recognition, the recognition rate of this method is up to 100%, and it can be used under occlusion of the faces. The method can handle some of the missing data. Ahdid et al [66] extracted the iso-geodesics on the three-dimensional face, then converted the curves to the shape space through the Square Root Velocity Function (SRVF), finally, calculated the distance between each pair of iso-geodesics for face recognition, and the highest recognition rate is up to 98.9%. Zhao et al. [67] proposed a method of 3D face similarity measurement based on geodesics in shape space for the large facial expression variations. Firstly, the method applied a group of geodesics (As shown in figure 13) to approximately represent the 3D face model which is pre-processed and registered. Then, the group of geodesics is transformed into the shape space, and their similarity is calculated by elastic metric. The similarity of the two 3D face models is calculated by using the average geodesic distance between the corresponding geodesics on the 3D face model under the shape space. This method combines the intrinsic attributes of geodesics with the advantages of elastic shape analysis suitable for large facial expression variations and can reflect well the similarity of the 3D face model.

This kind of method is robust for large facial expression variations with non-isometric. One drawback of this kind method is that it also cannot handle the problem of opening mouth. In this case, the assumptions are not true that the surface genus is zero. This kind of method requires that the facial surface is differentiable, so that their tangent space is also manifold. The elastic match of the facial curve can be accomplished by introducing a special elastic Riemannian metric under the shape space of various facial curves.

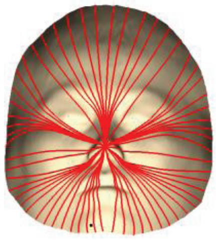

Figure 13. Geodesics on the 3D face [67]

5. **Conclusion and prospect.** In summary, the three-dimensional face recognition is a basic problem in computer graphics and computer vision. The 3D face recognition methods based on geodesics have had in-depth researches and good results in isometric deformation of 3D face, while the research on 3D face recognition methods based on geodesics under non-isometric deformation with large expression changes is just beginning. In the future, more face recognition methods under large expression variations, occlusions or missing face data are needed to be explored so that face recognition can provide important role for the similarity judgment, public security criminal investigation, confirmation of kinship and anthropological research. Thus it has prominent theoretical value and practical significance.

Acknowledgment. The authors gratefully appreciate the anonymous reviewers for all of their helpful comments. This work was funded by China Postdoctoral Science Foundation(2017M622137),the National Natural Science Foundation of China under Grant Nos. 61702293, 61772294 and 61572078, the Open Research Fund of the Ministry of Education Engineering Research Center of Virtual Reality Application(MEOBNUEVRA201601).

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